# Introduction to holography - Lecture IV

Last time : showed that, quantum-mechanically, black holes emit thermal rad.
@ the Hawking temperature $T_H = \frac{\hbar \kappa}{2\pi}$

$\Rightarrow$ laws of b.h. mechanics = laws of thermodyn, applied to b.hs.
and, in particular $S_{BH} = \frac{\mathcal{A}_H}{4G\hbar}$ is the entropy of the black hole.
(despite lack of immediat stat. int)

This time : discuss the profound implications of taking the Bekenstein-Hawking entropy seriously, as well as an entirely independent set of hints pointing in the same direction

- start discussing string theory

But first, a few comments on the 2nd law of black hole thermodyn.

- in his attempts to salvage the 2nd law of thermodyn in presence of black holes (which would otherwise become unverifiable), Bekenstein ('72) proposed to assign an entropy $S_{BH} = \eta \frac{\mathcal{A}_H}{\ell_p^2}$ ($\eta = 1/4 \Leftarrow$ Hawking's temp. calc.) to the black hole (! note no microscopic interpretation as state-counting !); it is now the generalised entropy

$$\boxed{S_{gen} = S_{out} + \frac{\mathcal{A}_H}{4G\hbar}}$$

that should increase in $\forall$ physical process : generalised 2nd law (GSL)

- motivated by Hawking's area thm (assumes null eng. cond. + cosmic censorship)
- note this is a classical theorem, & can be violated QM (e.g., in Hawking's evaporation process).

- the proper interpretation of $S_{out}$ is as entanglement entropy of (quantum) matter fields outside the black hole horizon (UV divergent). The sum appearing in $S_{gen}$ is better defined than each term separately (see e.g. gr-qc/9503003)

- the validity of the GSL requires a connection b/w the increase in horizon area (determined by the mass of the object that is thrown in) and the entropy of the matter system, which a priori seem unrelated $\Rightarrow$ Bekenstein bound $\simeq S_{matt} \leq \frac{2\pi}{\hbar} E_R$
  (! pure QFT restriction, as no $G_N$ appears; proper def$^n$ in terms of q. info.).
  Casini 2008

- $S_{BH} \propto A_H$ holds for black holes in Einstein gravity + matter. If higher curvature corrections are present, then the formula for black hole entropy needs to be appropriately modified ($\ni$ the 1$^{st}$ law still holds) $\Rightarrow$ Wald entropy $\oint_{\mathcal{B}}$ local, geom. quantity

- showing that $A_H/4G\hbar$ is the entropy of the black hole had to wait for major developments in string theory. First successful microscopic account of a black hole's entropy was in '96 (Strominger & Vafa) who reproduced it by counting states in a "dual" CFT description, where they could be easily identified & counted ($\nexists$ understanding for general black holes)

Obs : since $S \propto \frac{A_H}{\ell_p^2}$, it looks as if the d.o.f. of the b.h "live" on its surface, w/ $\sim 1$ d.o.f / Planck-size region ($S \sim A$ very few d.o.f. as compared to the standard $S \propto V$ in local th.)

- on the other hand, $A_H/\ell_p^2$ is huge. Exercise $S_{BH}(\odot)$

- since $S \propto A$ one would imagine that standard matter w/ $S \propto V$ could easily be more entropic than b.h.s (in a fixed spt. region). However, this is not the case, due to gravitational collapse (generic ← Penrose)

To see this, imagine a finite-temp (T) gas (described by standard QFT / CFT) in a region R of spt. (size R). At high T, we have

$$E_{gas} \sim T^4 V \sim T^4 R^3 \qquad S_{gas} \sim T^3 V \sim (TR)^3$$

- we would like that $S_{gas} > S_{BH}(R) = \frac{A_H}{4G} \approx \frac{R^2}{G}$ entropy of b.h. that fills region R

- on the other hand, the eng. of the gas in R is limited by

$$GE < R \qquad \text{(lest it undergo grav. collapse)}$$

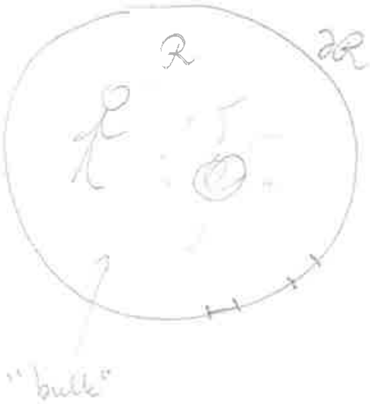
$$\Rightarrow \quad T < \left(\frac{1}{R^2 G}\right)^{1/4} \quad \Rightarrow \quad S_{gas} < \left(\frac{R^2}{G}\right)^{3/4} \sim \left(\frac{A_H}{G}\right)^{3/4} \ll \frac{A_H}{G} \quad \text{for } R \gg \ell_p.$$

⇒ the gas cannot have larger entropy than a b.h. filling the region

The far-reaching implication of this apparently simple obs

$$S_{\psi \text{ in } R} < S_{BH}(R) = \frac{\text{Area}}{4G\hbar}$$

is that we can encode all that happens $\subset R$ by d.o.f. that live on the surface of region $R$ ⇒ HOLOGRAPHY ('t Hooft '93)




- the QG d.o.f live in fewer dims than the cls. limit indicates
- can encode th. w/ gravity in $R$ by th. w/o grav. in $\partial R$
- bulk locality: approximate @ best, spt → emergent
- this arg. is very general, but also v. vague

This time: will present a $\neq$ arg, also due to 't Hooft, that a large $N$ gauge th (or, more generally, a th. w/ matrix d.o.f.) is connected to a string th. (usually contains gravity) $\approx$ emergent gravity descr.

- start discussing pert. strings prop. on flat space

## The large $N$ limit of gauge th.

- consider $SU(N)$ Yang-Mills th., w/ $N = \#$ of colours
- the gauge fields $A_\mu^a$ correspond to the $N^2-1$ generators of $SU(N)$
- they can be repr. as hermitean $N \times N$ matrices $(A_\mu)^i{}_j = \sum_{a=1}^{N^2-1} A_\mu^a \underbrace{(T_a)^i{}_j}_{\text{gen. of the fund rep}}$

where the $1^{st}$ (upper) index transf. under the fund. & the $2^{nd}$ (lower) -11- anti-fund.

$\delta_a A^b = -f_{ac}{}^b A^c$ $\qquad$ $\delta_a A^i{}_j = (\delta_a A^b)(T_b)^i{}_j = -f_{ac}{}^b A^c (T_b)^i{}_j \overset{\text{o/e}}{=} i[T_a, T_c]^i{}_j A^c$

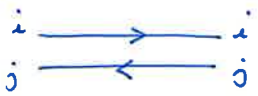
$[T_a, T_b] = i f_{ab}{}^c T_c$ $\qquad$ $= i \underbrace{T_a{}^i{}_k}_{\text{fund}} (A)^k{}_j - i (A)^i{}_k (T_a)^k{}_j \quad -(\bar{T}_a)_j{}^k$

- the Lagr. is $\mathcal{L} = -\frac{1}{4}\,\mathrm{tr}\,F^2$ $\quad (F_{\mu\nu})^i{}_j = \partial_\mu A_\nu{}^i{}_j - \partial_\nu A_\mu{}^i{}_j + g[A_\mu, A_\nu]^i{}_j$

- will be useful to work in terms of $A' \sim gA \Rightarrow F' = dA' + [A', A'] = gF$
  & $\mathcal{L} = -\frac{1}{4g^2}\,\mathrm{tr}\,F'^2$

- Feynman rules $\quad {}^i{}_j \sim\!\sim\!\sim\!\sim {}^k{}_l \ \propto g^2\, \delta^{ik}\delta_{jl}$
  (ignoring all Lorentz & momentum indices, we are only interested in dependence on $N$ & $g$)

- useful to introduce the double line notation



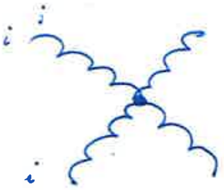

$\times g^2$

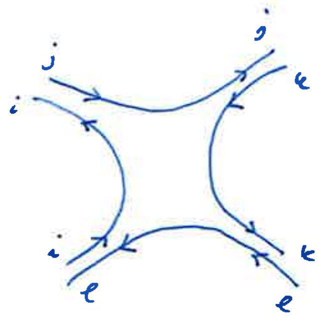
propagator

(upper index: outgoing; lower: incoming)

- interactions

$\frac{1}{g^2}$



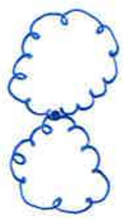

$\times \frac{1}{g^2}$

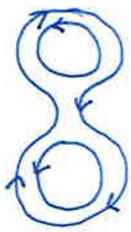
+ 3-pt. vertex

- each index loop $\rightarrow \sum_i \rightarrow N$
  (not # loops in Feynman diag!)

- Feynman diag. will look like

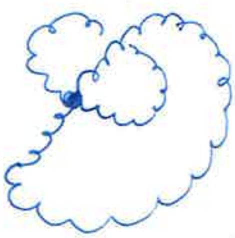
1.

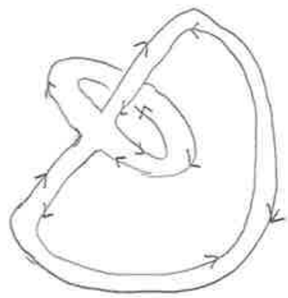




$\frac{1}{g^2}(g^2)^2 N^3 \sim N^3 g^2$

2.

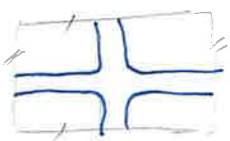


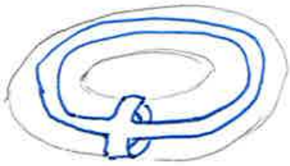


$\frac{1}{g^2}(g^2)^2 N \sim N g^2$

- the 1st diag. is called planar (can be drawn on a plane)

- the 2nd cannot be drawn on a plane, but can be straightened out on a torus ($\ni$ it doesn't self-cross)





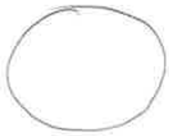
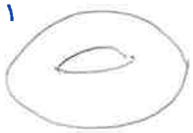
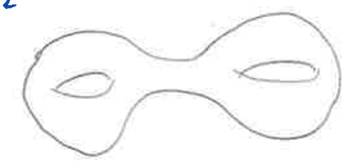
- note that power of $N$ = # faces after straightening

- what is the general $N$-counting / classification of diagrams?

- A Feynman diag. w/ $E$ propagators, $V$ vertices and $F$ loops can be straightened out on a (compact) 2d surface of genus $g$ (# holes in the surface), w/

$$2-2g \equiv \underbrace{\chi}_{} = F+V-E$$

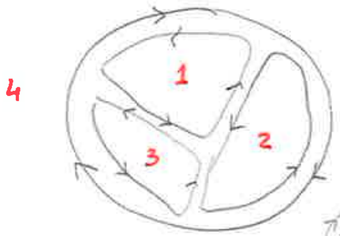
Euler characteristic (topological invar. of 2d surfaces $\sim \int R\sqrt{g}$)

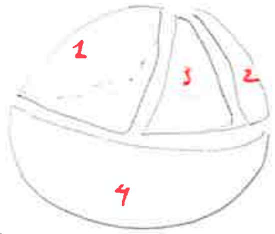
- $g=0$  $g=1$  $g=2$ 

(add point @ $\infty$)

- the Feynman diagram corresp. to a partition of the surface that separates it into polygons ($\approx$ "triangulation)

e.g.




$F=4$
$V=4$
$E=6$

$\chi=2 \Rightarrow g=0$.

$F=3$
$V=1$
$E=2$

where an edge is defined to connect b/w 2 vertices (or 1 vertex w/ itself)

- the $g,N$ dependence of such a diagram is $\mathcal{A} \sim (g^2)^{E-V} N^F$

Note though that $E=1$, $V=1$; $E=3$, $V=3$ } same $E-V$

- since the # of loops ($F$) can be arbitrarily large, it does not look like $\exists$ good large $N$ expansion

- however, 't Hooft noticed that taking $N$ large w/ $\lambda \equiv g^2N$ fixed ('t Hooft limit)
then $\mathcal{A} \sim (g^2N)^{E-V} N^{\overbrace{F+V-E}^{\chi=2-2g\,\leq\,2}}$ does have one

- the expansion is in terms of the topology of the Riemann surf. assoc. to the Feynman diagram, & takes the general form

$$\mathcal{A} = \sum_{g=0}^{\infty} N^{2-2g} \sum_{n=0}^{\infty} c_{g,n} \lambda^n \quad = N^2 f_0(\lambda) + N^0 f_1(\lambda) + \frac{1}{N^2} f_2(\lambda) + \dots$$

– good large $N$ expansion (low genus dominates)

- as $N \to \infty$, only planar diagrams survive (large simplification, but still an $\infty$ # of them!)

- for large $\lambda$, diagrams w/ a large # of loops (faces) dominate as the lines become dense $\to$ smooth 2d surface $\approx$ worldsheet of a string

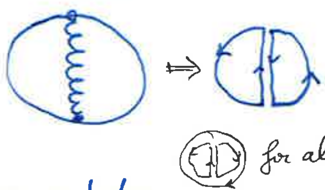
interestingly, perturbative string th. is also organised into a topological expansion involving 2d surfaces, & string. th. is known to contain gravity

large $N$ gauge th ('t Hooft limit) $\approx\!>$ strings? $\approx\!>$ gravity?

Q: Can this be made more precise?

- similar results hold if we include matter transf. in the fundamental repr. (quarks)
  - same counting for vertices & prop. after rescaling
  - quark / anti-quark : single line : $\longrightarrow$ :
  - for the same "triangulation", one factor of $N$ less for each quark loop. $\Rightarrow \mathcal{A} \sim N^{2-2g-L}(g^2N)^{E-V}$

Ex:



$\Rightarrow$ vs

for all gluons

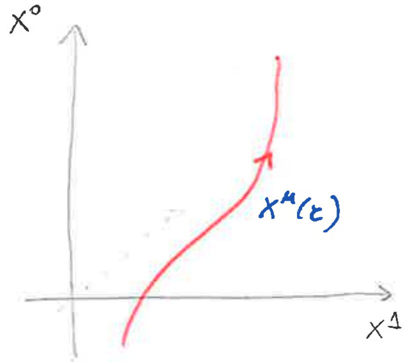
$L$ = # quark loops = holes

- this large $N$ counting applies to $\forall$ model w/ matrix-valued fields

# String theory

- starting assumption: basic objects are strings, rather than point part.*

- their action can be understood via an analogy w/ the relativistic point part. we have already discussed




  - relativistic point part. travelling in some D-dim'l spt. w/ coord $X^\mu$ $\quad \mu = \{0, \dots, D-1\}$

  - traces worldline $X^\mu(\tau)$
    ↑ arbitrary worldline param.

  - action: length of the space-time interval $\quad ds^2 = \underbrace{G_{\mu\nu}(x)}_{\eta_{\mu\nu}\text{ for now}} \dot{X}^\mu \dot{X}^\nu \qquad \dot{} = \frac{d}{d\tau}$

$$S = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu}$$

classical traj: geodesics

  - manifest target space Lorentz invar

  - reparametrization invariant $\quad \tau \to \tau' = f(\tau) \quad \Rightarrow$ constraints

$p^\mu = \frac{m\dot{X}^\mu}{\sqrt{-\dot{X}^2}}$ sat. $\underbrace{p^2 + m^2}_{\propto H} = 0$

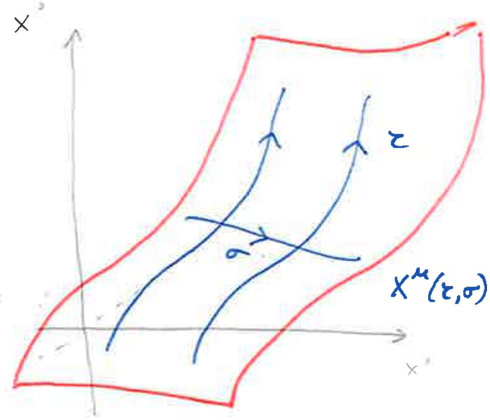
- a string is an object w/ 1 spatial dim. ⇒ traces a 2d worldsheet as it moves throughout spt. (target sp.)

$X^\mu(\underbrace{\tau, \sigma}_{\text{arbitrary w-sheet param}})$ $\qquad \mu = \{0, \dots, D-1\}$

$\tau \in \mathbb{R}$ , $\quad \sigma \in (0, \pi)$ open string

$\sigma \in (0, 2\pi)$ closed string

$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$




target sp. metric $\eta_{\mu\nu}$

- induced metric on the string = pullback of the target sp. metric

$$h_{ab} = \eta_{\mu\nu}\, \partial_a X^\mu \partial_b X^\nu \qquad \sigma^a = \{\tau, \sigma\} \quad a = \{0,1\} \qquad h_{ab} = \begin{pmatrix} \dot{X}_\mu \dot{X}^\mu & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'\cdot X' \end{pmatrix}$$

$\dot{} = \frac{d}{d\tau}$ $\quad {}' = \frac{d}{d\sigma}$

- bosonic string action : area of the string w-sheet (pt. part. analogy)

$$\underbrace{S_{NG}}_{\text{Nambu-Goto}}[X^\mu] = -\underbrace{\frac{1}{2\pi\alpha'}}_{\text{string tension}} \int d^2\sigma \sqrt{-\det h_{ab}}$$

reparam. invar.

Lorentz invar

string tension $\quad [T] = (\text{length})^{-2}$

string length : $l_s$ w/ $\alpha' = l_s^2$

- the string tension can be interpreted as its energy / unit length

  - note NG action takes the form $S_{NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma \sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 (X')^2}$

  - in static gauge, $\tau = X^0/R \leftarrow$ dimensionful ct. (drops out)

  - consider snapshot of string when $\frac{d\vec{X}}{dt} = 0$ (no kinetic eng). Then $\dot{X}^\mu = \{R, 0, 0, \ldots\}$

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d\sigma \cdot \frac{dt}{R} \sqrt{\left|\frac{d\vec{X}}{d\sigma}\right|^2 R^2} = -\frac{1}{2\pi\alpha'}\int dt \cdot \text{length of string}$$

$\Rightarrow$ potential energy $\quad V = \frac{1}{2\pi\alpha'} \times \text{length string}$

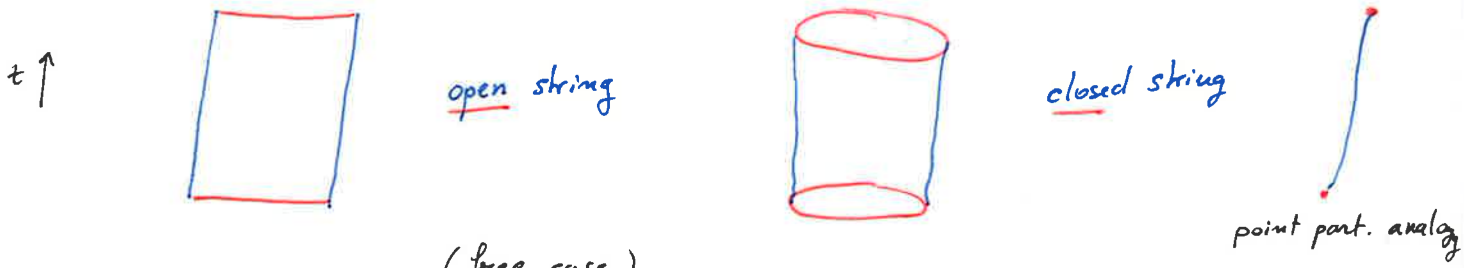
- $\exists$ another, classically equivalent, form of the action that is more useful when attempting to quantize this system using path integral techniques: Polyakov action

$$S_P[X^\mu, \underbrace{\gamma_{ab}}_{\text{worldsheet metric}}] = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{\gamma}\, \gamma^{ab} \underbrace{\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}}_{h_{ab}}$$

- $\gamma_{ab}$ e.o.m $\quad T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\gamma_{ab}\, \partial^c X^\mu \partial_c X_\mu = 0 \quad \rightarrow$ solve for $\gamma_{ab}$ & plug into $S_P$ to get $S_{NG}$.

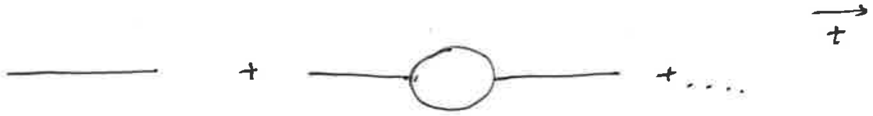
- before getting into the details of quantization, let us discuss the structure of string perturbation th.

- the amplitude for a string to propagate from some initial to some final state is given by the path integral $\int \mathcal{D}X^\mu \mathcal{D}\gamma \, e^{iS_P}$ w/ fixed initial/final curves

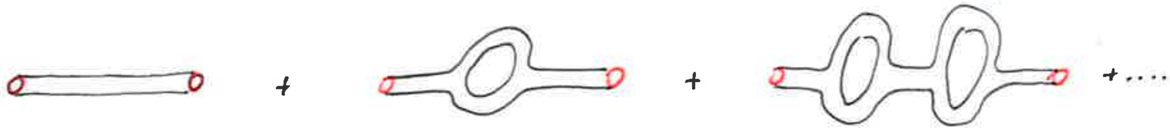



(free case)

- to include perturbative interactions in QFT we sum up Feynman diagrams

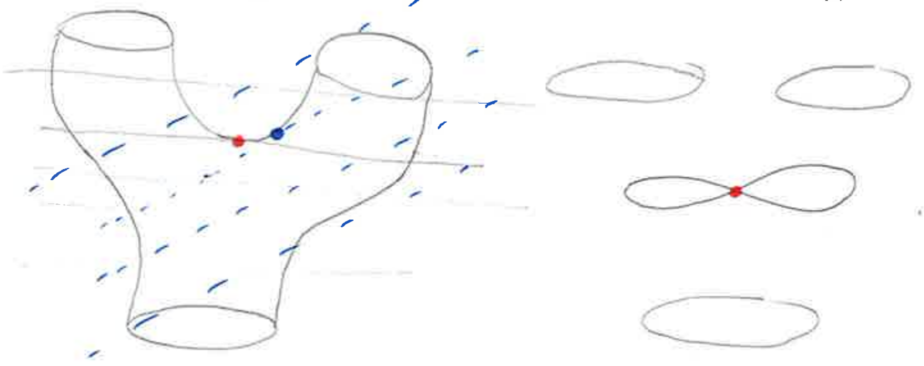



- in string theory, the prescription is to sum up smooth 2d surfaces




  - locally, the worldsheet is the same as in the free case
  - interactions only arise from the global topology of the w-sheet

- reason: in string th, the interaction does not happen @ a specified pt. in spt.



string splitting interaction.

≠ Lorentz obs see int. happening @ ≠ points

(no Lorentz-invar contact interaction)

- smoothing out $\approx>$ no UV divergences in str. th.

- one may add a weighing factor for $\neq$ topologies by letting

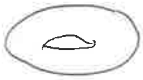
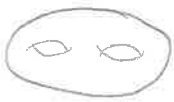
will discuss its int later

$$S_{string} = S_P + \lambda \chi$$

Euler characteristic of the surface

$$\chi = \frac{1}{4\pi}\int_\Sigma d^2\sigma \sqrt{\gamma}\, R[\gamma] + \frac{1}{2\pi}\int_{\partial\Sigma} k\, ds$$

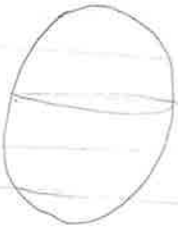
- then, for e.g. vacuum processes we have

$Z_{string} = e^{-2\lambda} \times$  $+$  $+ e^{2\lambda} \times$  $+ \dots$

same genus expansion as that we found in large N gauge th!

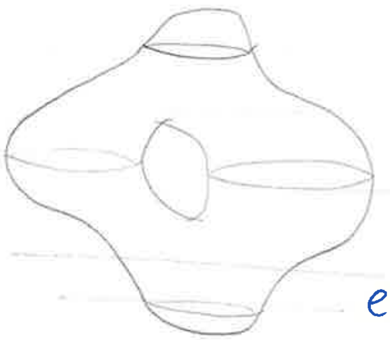
- $e^\lambda$ : interpreted as amplitude for emitting a closed string $\sim g_s$

string coupling



$e^{-2\lambda}$

string nucleates from & gets reabsorbed into vacuum



$e^0 = e^{-2\lambda} \cdot e^{2\lambda}$ $\begin{cases} e^\lambda & \text{reabsorb} \\ e^\lambda & \text{emit} \end{cases}$

string emerges from vacuum, spits out a string, reabsorbs it & disappears into vac

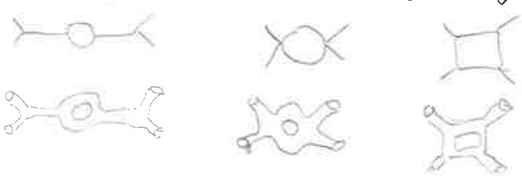
- similarly, for open strings, adding a hole $\Rightarrow \Delta\chi = -1$  $\sim e^\lambda$

$\Rightarrow$ amplitude for emitting / abs. an open string is $e^{\lambda/2}$

- thus, string pert. th $\rightarrow$ classification of euclidean 2d surfaces (genus g)

- in a certain limit, one may think of string "Feynman" diag. as just fattened QFT diag

note, however, that # string diag @ given loop order $\ll$ # QFT Feynman diag, e.g.



same topology, $\neq$ param