Introduction to holography - Lecture II

Last time : Hamiltonian formulation of GR

- ADM Hamiltonian is a pure boundary term on-shell
  ← hint of holography
- energy = boundary integral

This time : much stronger hint of holography in classical GR

Black hole thermodynamics

- very deep & fundamental relationship betw/ gravity, thermodyn & the quantum theory; geometric

$$S_{BH} = \frac{A_H}{4G\hbar}$$

- to gain insight into the very peculiar behaviour of black holes we will follow the historical development of the subject, starting w/ the seminal work of Penrose on the inevitable formation of singularities in gravitational collapse

- at the time (60's), it was proposed / believed that singularity ⇐ artifact of spherical symm
  ⇒ generic configs would be nonsingular

- Penrose → singularities form generically (+ cosmic censorship → black holes generically form)
  ⇒ black holes are ubiquitous

  → introduced novel concepts (trapped surface) & techniques
  – horizons treated as dynamical entities

- Plan : discuss trapped surfaces & singularity thms $\xrightarrow{\text{Raych.}}$ area thm

Gravitational collapse

- consider the spherically symmetric collapse of a star

- by Birkhoff's thm (spherical symm + Einstein eqns => staticity) the metric outside the surface of the star is the Schwarzschild metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega^2$$

- if $M \gtrsim 3 M_\odot$, the collapse cannot be halted by the degeneracy pressure of electrons / neutrons => black hole w/ horizon @ $r = 2M$

- before continuing, let us remember the definition of a black hole

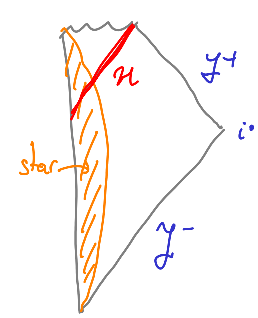
Black hole

- intuitively, a black hole is a region of spt. where gravity is so strong that nothing can escape, even light

- we however need to specify: escape to where? E.g. for AF spt.

- the black hole region, $\mathcal{B}$, of an asymptotically flat space-time, $\mathcal{M}$, is defined as

$$\mathcal{B} \equiv \mathcal{M} - \underbrace{J^-}_{\text{causal past}}(\mathscr{I}^+)$$



- the (future) event horizon $\mathcal{H} \equiv \partial\mathcal{B}$

  null hypersurface by construction

- note that $\mathcal{H}$ has no distinguished local significance: the entire future history of the space-time must be known in order to determine its location

- since the Schwarzschild coord. system breaks down @ the horizon, (lightcones close up) introduce Eddington-Finkelstein coord. instead

$r_* =$ tortoise coord $\left(dr_* = \dfrac{dr}{1-\frac{2M}{r}}\right)$

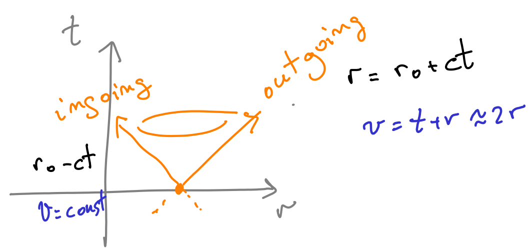
adv. EF time

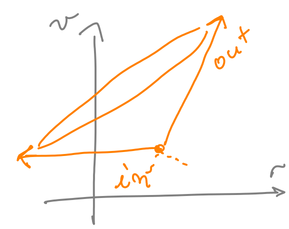
$$v = t + r_* = t + r + 2M\ln(r-2M)$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)\left((dv-dr_*)^2 - dr_*^2\right) + r^2 d\Omega_2^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega_2^2$$

- metric smooth @ $r=2M$; curvature singularity @ $r=0$ in full Schw. spt.

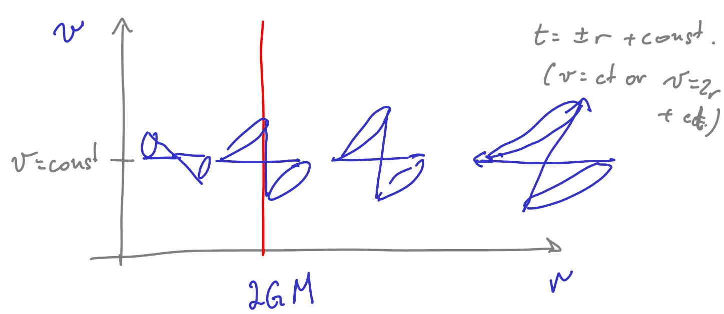
- lightcones Minkowski





- for $r<2M$, the lightcones "tilt" → look @ radial null curves (θ, φ const)

$$\begin{cases} \text{'ingoing'}: & dv = 0 \\ \text{'outgoing'}: & \dfrac{dv}{dr} = \dfrac{2}{1-\frac{2M}{r}} < 0 \end{cases}$$

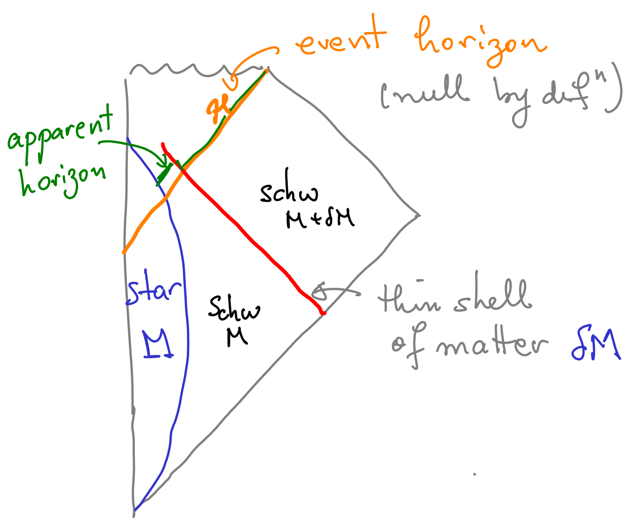


⇒ all causal curves go to $r=0$ in finite λ

- intuitively, the gravitational field due to matter focusses lightrays, and for $r<2M$ the effect is so large that all lightrays are convergent (the area of the lightfront is decreasing) (to reach ∞, it would need to increase)

• this intuition leads to Penrose's notion of trapped surfaces inside the horizon: compact, codimension 2 smooth spacelike manifolds w/ the property that both sets (i.e., ingoing & outgoing) of future-directed null geodesics are converging

• thus, a local manifestation of the presence of a horizon is the presence of trapped surfaces. The boundary of the region $S(t)$ (~time coord) containing trapped surfaces is called the apparent horizon, whose location can be determined locally by knowing the metric on $S(t)$. This is not true of the event horizon, for which all future history must be known to determine its location



• note that the notion of a trapped surface does not rely on spherical symm.

• their presence indicates the presence of a singularity (Penrose), as we now discuss

• in order to describe them more formally we first need to discuss geodesic congruences and the Raychoudhuri eqn used in the proof of singularity thms.

• we will discuss congruences of timelike geodesics, then quote the results for null congruences that are needed in the b.h. discussion

Geodesic congruences

- set of curves in an open region $\mathcal{O}$ of spt. $\ni$ each pt. $\in \mathcal{O}$ lies on precisely one curve (no intersections)

- tangents to a congruence $\leftrightarrow$ vector field in $\mathcal{O}$

- consider first a congruence of (affinely param.) timelike geodesics, w/ tg. $\xi^\mu = \frac{dx^\mu}{d\tau}$, $\xi_\mu \xi^\mu = -1$

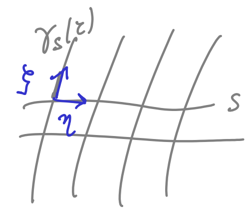
- then the tensor field $B_{\mu\nu} \equiv \nabla_\nu \xi_\mu$ is purely 'spatial', ie.

$$\underbrace{\xi^\mu B_{\mu\nu}}_{\xi^2=-1} = \underbrace{B_{\mu\nu}\xi^\nu}_{\text{geod. eqn}} = 0$$

- to understand the physical interpretation of $B_{\mu\nu}$, consider the orthogonal deviation vector η^μ, representing the infinitesimal displacement vector $\frac{\partial}{\partial s}$ between nearby geodesics in a one-parameter sub-family of geodesics $\gamma_s(\tau)$ w/ integral curves $x^\mu(\tau, s)$ (s labels geodesics).

  The vect. fields $\xi^\mu = \partial_\tau x^\mu$ & $\eta^\mu = \partial_s x^\mu$ commute ($\partial_s \xi^\mu = \partial_\tau \eta^\mu$)

$$\underset{\text{covar.}}{\Rightarrow} \quad \xi^\mu \nabla_\mu \eta^\nu = \eta^\mu \nabla_\mu \xi^\nu = \underbrace{B^\nu{}_\mu}\, \eta^\mu$$



$B^\nu{}_\mu$ measures failure of η to be $\parallel$-transported

- define the "spatial metric" $h_{\mu\nu} = g_{\mu\nu} + \xi_\mu \xi_\nu$ = projection operator onto subspace $\perp \xi$

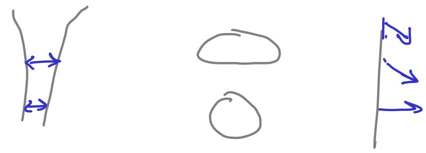
  - note $B_{\mu\nu}$ spatial : $B_{\mu\nu} = h_\mu{}^\rho h_\nu{}^\sigma B_{\rho\sigma}$ (proj.)

- define : expansion $\quad \theta \equiv B^{\mu\nu} h_{\mu\nu} = \nabla_\mu \xi^\mu$ (measures average expansion of nearby geod.)

  $\theta = \frac{1}{2} h^{\mu\nu} \mathcal{L}_\xi g_{\mu\nu} = \frac{1}{2} h^{\mu\nu} \mathcal{L}_\xi h_{\mu\nu} = \partial_\tau \sqrt{\det h}$

  shear $\quad \sigma_{\mu\nu} = B_{(\mu\nu)} - \frac{1}{3}\theta h_{\mu\nu}$ ← purely spatial $\sigma_{\mu\nu}\xi^\mu = 0$.

  twist $\quad \omega_{\mu\nu} = B_{[\mu\nu]}$ $\qquad$ of the congruence

- so $B_{\mu\nu}$ can be decomposed as 

$$B_{\mu\nu} = \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}$$

- the rates of change w/ τ of θ, $\sigma_{\mu\nu}$, $B_{\mu\nu}$ are determined by their defn

- only interested in $\frac{d\theta}{d\tau} = \xi^\mu \nabla_\mu \theta = \xi^\mu \nabla_\mu \nabla_\nu \xi^\nu = \xi^\mu\left(\nabla_\nu \nabla_\mu \xi^\nu + R^\nu{}_{\lambda\mu\nu}\xi^\lambda\right)$

$$= -R_{\mu\lambda}\xi^\mu\xi^\lambda + \nabla_\nu\big(\underbrace{\xi^\mu\nabla_\mu\xi^\nu}_{0}\big) - \nabla_\mu\xi^\nu\nabla_\nu\xi^\mu = -\underbrace{B_\mu{}^\nu B_\nu{}^\mu}_{\frac{1}{3}\theta^2 + \sigma_{\mu\nu}\sigma^{\mu\nu} - \omega_{\mu\nu}\omega^{\mu\nu}} - R_{\mu\nu}\xi^\mu\xi^\nu$$

=> Raychaudhuri's eqn

$$\boxed{\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}\xi^\mu\xi^\nu}$$

- purely geometrical eqn.

- widely used in proving singularity thms, using Einstein's eqn. to trade $R_{\mu\nu}$ for $T^{\text{matter}}_{\mu\nu}$, which generally obeys energy conditions

Cosmological singularity theorems

- assume $\omega_{\mu\nu} = 0 \Leftrightarrow$ the congruence is hypersurface orthogonal, i.e. $\xi_\mu \propto \partial_\mu f$

- assume the matter fields satisfy the strong energy condition (non-rel matter, rad, but not $\Lambda > 0$!)

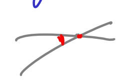
$$T_{\mu\nu}\xi^\mu\xi^\nu \geq +\frac{1}{2}T\xi^2, \qquad \forall\, \xi^\mu \text{ timelike}$$

- then $R_{\mu\nu}\xi^\mu\xi^\nu \overset{d=4}{=} \left(G_{\mu\nu} - \frac{1}{2}\,\mathrm{tr}G\, g_{\mu\nu}\right)\xi^\mu\xi^\nu = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right)\xi^\mu\xi^\nu \geq 0$

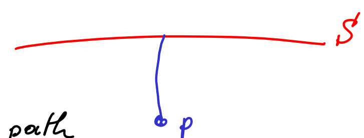
- and therefore $\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0 \quad \Rightarrow \quad d\left(\frac{1}{\theta}\right) \geq \frac{d\tau}{3} \quad \leadsto \quad \frac{1}{\theta_f} - \frac{1}{\theta_i} \geq \frac{\tau}{3}$

- then, if θ takes a negative value $\theta_0 < 0$ @ $\tau = 0$, then $\theta \to -\infty$ along that geodesic within proper time $\tau \leq \frac{3}{|\theta_0|}$

- similarly, if θ — positive — $\theta_0 > 0$, then $\theta \to \infty$ along that geodesic towards the past, no longer that $\tau = \frac{3}{\theta_0}$ before. (θ larger in the past)

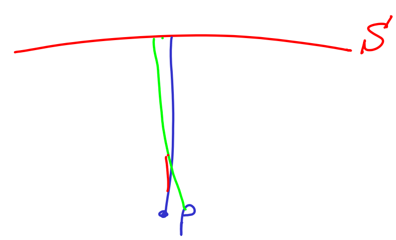
- however, note that $\theta = \frac{\dot{\mathcal{A}}}{\mathcal{A}} \to \pm\infty$, could either represent a spt. singularity ($\dot{\mathcal{A}} \to \infty$), or a singularity in the congruence ($\mathcal{A} \to 0$), i.e. a focal point 

- to exclude the latter possibility, Hawking assumed spt. is globally hyperbolic w/ initial surface S (every causal path through $p \cap S$)



- as a conseq. every point p is connected to S by a causal path of maximum proper time $\to$ a timelike geodesic $\perp S$

- such a geodesic cannot have focal points: if $\exists$ another nearby causal path that $\cap$ it, one can reduce the length by rounding off the corner $\Rightarrow$ not length-minimizing



$\Rightarrow \nexists$ no point in spt. that is to the past of S a proper time $> 3/\theta_{min}$, where θ_{min} is the minimum value of θ_0 on the initial slice $S \Rightarrow$ Big Bang

- for applications to black holes, need to discuss instead null geodesic congruences (more complicated b/c tg. vector is null, so $\perp$ vect. only defined up to $\eta^\mu \to \eta^\mu + \alpha \underbrace{k^\mu}_{\text{tg. vect.}}$)

- Raychaudhuri's eqn. for null geodesics

$$\boxed{\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{\mu\nu}\hat{\sigma}^{\mu\nu} + \hat{\omega}_{\mu\nu}\hat{\omega}^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu}$$

- the area of an infinitesimal surface element $\perp$ lightfront changes as $\delta A = A\theta\,\delta\lambda$ as one moves a distance $\delta\lambda$ along the congruence.

Exercise: Assume matter satisfies the null energy condition ($T_{\mu\nu}k^\mu k^\nu \geq 0$, $\forall\, k^\mu$ null). If $\theta = \theta_0 < 0$ @ any point along a geodesic, then $\theta \to -\infty$ along that geodesic within finite affine parameter $\lambda \leq 2/|\theta_0|$.

- we will again assume that $\hat{\omega}_{\mu\nu} = 0$

Penrose's singularity thm

- trapped surface: compact, codimension 2 smooth spacelike manifold T w/ the property that the expansion θ of both sets (i.e., ingoing & outgoing) of future-directed null geodesics is everywhere negative.

- suppose: $\mathcal{M}$ is a connected, globally hyperbolic spt w/ a noncompact Cauchy surf Σ

  - $R_{\mu\nu}k^\mu k^\nu \geq 0$ $\forall\, k^\mu$ null ($\Leftarrow$ e.g. if Einst. eqn. + null eng. cond)

  - $\mathcal{M}$ contains a trapped surface T, w/ $\theta_0 < 0$ being the max of θ for both sets of orthogonal geodesics on T

- then, at least one inextendible future-directed orthogonal null geodesic from T has affine length no greater than $2/|\theta_0|$

- crux of the proof: show that the singularity in θ is due to a sing. in spt, rather than just in the congruence → null analogue of the maximum proper time of timelike geodesics

    (promptness → no other causal path could have arrived sooner

    - look @ the bnd. of the future of T (see e.g. Witten's PITP lectures 2018)

⇒ at least one null geod. cannot be extended past its 1^{st} focal pct ⇒ geod. incompleteness

- the likely / assumed reason behind this is the presence of a spt. singularity, as in Schw.

Comments:

- even though Penrose's derivation says nothing about the nature of the singularity it is important b/c:

    - it devises a local criterion (trapped surf) for the eventual formation of a sing.

    - since the formation of trapped surfaces is generic in gravitational collapse ⇒ singularities are generic (not an artifact of spherical symm., as previously believed)

- are there singularities inside or outside horizons? Cosmic censorship conjecture: inside, to save predictability of GR ⇒ collapse generically → black holes

- If black holes form generically from gravitational collapse, what are the most general final equilibrium states? (Carter p. 129)

- such states $\nexists$ for matter if $M > 3M_\odot$

- Israel '67 staticity => spherical symmetry => b.h. sols are unique, up to M, J
  (also: stationarity => axisymmetry)

- "Black holes have no hair" (Wheeler) => no d.o.f. except those that cannot be radiated away

  (true for electrovac. b.h. in 4d)

- final state of gravitational collapse is a b.h. + b.h are unique $\mapsto$ what happens to entropy of matter thrown in? → beginning of black hole thermodynamics

- to give a hint, the focusing eqn. can also be used to show that the expansion θ of the horizon generators (null geodesics whose tg. vector = normal to $\mathcal{H}$) must be positive (≥ 0), assuming the null energy condition.

- indeed, if $\theta < 0$ somewhere, then $\theta \to -\infty$ within finite affine param => naked singularity on the horizon => $\theta \geq 0$. Thus, in Einstein gravity w/ $T_{\mu\nu}k^\mu k^\nu \geq 0$ the area of the horizon always increases.

- this resembles the 2nd law of thermodyn, that entropy does not decrease