Introduction to holography - Lecture XII

Last time: match spectrum pert. of $AdS_{d+1} \times S^\sigma$ w/ the CFT one.

bdnd theory	bulk theory
· conformal family w/ primary Δ	· bulk field w/ $m^2 \ell^2 = \Delta(\Delta - d)$ (match quadratic Casimir).
· integer-spaced spectrum $\Delta + n \in \mathbb{Z}$	· energy scalar field in AdS (norm. @ ∞ + smoothness in int) $\Rightarrow \Delta = \Delta + n$
· large N factorization (single-trace ops) GFF	· approximately free bulk field (int. perturbative in $\frac{1}{N}$)
· multitrace ops $\lvert 0 \rangle$	· multiparticle states in AdS. EFT.

This time: the holographic dictionary

- holographic computation of corr. f.

The holographic dictionary for a free massive scalar field

- $S = -\frac{1}{2}\int d^{d+1}x\,\sqrt{-g}\,(g^{MN}\partial_M\phi\,\partial_N\phi + m^2\phi) + \mathcal{O}(\frac{1}{N})$ usually obtained via KK red.

- $ds^2 = \ell^2\,\frac{\eta_{\mu\nu}dx^\mu dx^\nu + dz^2}{z^2}$ Poincaré coord on AdS_{d+1}, $z \to 0$ bnd.

- the e.o.m. takes the form $(z^2\partial_{z^2}^2 + z^2\Box_\eta + (1-d)z\partial_z - m^2\ell^2)\phi(z,x^\mu) = 0$

- 2^{nd} order eqn $\Rightarrow$ 2 indep. sols. Assuming $\phi \sim z^\Delta f(x^\mu)$ as $z \to 0$, we obt.

$$\Delta(\Delta-d) = m^2\ell^2 \Rightarrow \Delta_\pm = \frac{d}{2} \pm \underbrace{\sqrt{\frac{d^2}{4} + m^2\ell^2}}_{\nu}$$

$\in \mathbb{R}$ if $m^2\ell^2 \geq -\frac{d^2}{4}$ BF bound

$\Delta_+ = \Delta$

$\Delta_- = d - \Delta$

- thus, the near-bnd. exp. of the scalar field takes the form

indep from p.d.v. of near-bnd. analysis, however they get rel. via interior

general sol'n: (general Δ)

$$\phi(z,x^\mu) = \underbrace{\phi_0(x^\mu)\,z^{d-\Delta}}_{\text{(usually) non-normalizable w.r.t. KG norm}} + \underbrace{\cdots\cdots}_{\substack{\text{subleading} \\ (\propto z^{2n})\text{ terms} \\ \propto \Box^n\phi_0}} + \underbrace{\phi_\Delta(x^\mu)\,z^\Delta}_{\text{normalizable}} + \underbrace{\cdots}_{\substack{\text{subleading terms} \\ \Box\phi_\Delta}}$$

∞ expansion

- more precisely $\langle\phi_1,\phi_2\rangle = -i\int_\Sigma dz\, d^{d-1}x\, \underset{\frac{1}{z^d}}{\sqrt{\gamma}}\, \overset{z}{n^\mu}\left(\phi_1^*\partial_\mu\phi_2 - \phi_2^*\partial_\mu\phi_1\right) \sim \int z^{\overbrace{2\Delta_\pm}^{d\pm2\nu}}\frac{dz}{z^{d-1}}$

$\Rightarrow$ z^{Δ_+} is always normalizeable

see Klebanov & Witten 9905104

- $z^{\Delta_-} = z^{d-\Delta} \sim z^{\frac{d}{2}-\nu}$ is normalizeable if $\nu \in [0,1)$

  - if $\nu > 1$, then norm: $\Rightarrow \phi_0 = 0$. ($z^{d-\Delta}$ mode not norm.)
  - if $\nu \in (0,1)$, then either $\begin{cases} \phi_0 = 0 \text{ or fixed} \\ \phi_\Delta = 0 \text{ (fixed)} \end{cases}$ standard quantiz $\Delta_+ \subset$ spectr / alternate quantiz $\Delta_- \subset$ spectr

how to get ops w/ $\frac{d-2}{2} < \Delta < \frac{d}{2}$

- from now on, we will only consider standard quantiz.

- normalizable modes are used to build up the bulk Hilbert sp & , in part, are allowed to fluctuate.

- non-normalizable modes are not part of the bulk Hilbert space. They are thus not allowed to fluctuate, but must instead be fixed as boundary cond.

  - more precisely, @ the level of the full bulk field we require that

$$\phi(x^\mu,z)\Big|_{z=\varepsilon} = \varepsilon^{d-\Delta}\phi_0(x^\mu) \quad \text{as} \quad \varepsilon\to 0.$$

"renormalized" bnd. cond. ϕ_0 finite as $\varepsilon \to 0$

- note that under a rescaling of the CFT coord, $x^\mu \to x'^\mu = \lambda x^\mu$, which in AdS corresp. to the isometry $x^\mu \to \lambda x^\mu$, $z \to \lambda z$, $\phi(x^\mu,z)$ is inert (scalar) $\Rightarrow$

$$\underbrace{\phi_0(x^\mu) = \lambda^{d-\Delta}\phi_0'(x^{\mu\prime})}_{\text{same as source for } O_\Delta} \qquad \underbrace{\phi_\Delta(x')\lambda^\Delta = \phi_\Delta(x)}_{\text{same transf. as operators of dim } \Delta,\ \mathcal{O}_\Delta}$$

$$\int d^dx\ \phi_0(x^\mu)\, \mathcal{O}_\Delta(x^\mu)$$

- note that $\Delta = d$ (marginal op) $\Leftrightarrow$ $m = 0$ (bulk field)

  $\Delta < d$ (relevant op) $\Leftrightarrow$ $m^2 < 0$, & for $\Delta < \frac{d}{2}$ they need to be realised via alternate quantiz.

  $\Delta > d$ irrelevant $\Leftrightarrow$ $m^2 > 0$.

- the source terms are usually turned on $\frac{1}{\infty}$ to compute corr. f.
  - for relevant ops, can also turn on the soureed a finite amount (fixed)

    $\Rightarrow$ holographic RG flows

- we are now ready to state the AdS/CFT dict. as an equality of part. f.

$$\boxed{Z_{AdS_{d+1}}\left[\phi(z,x^\mu)\Big|_{z\to 0} = z^{d-\Delta}\phi_0\right] = e^{W_{CFT}[\phi_0]} = \left\langle e^{\int d^dx\, \phi_0(x)\mathcal{O}(x)}\right\rangle_{CFT}}$$

↗ bulk partition f. w/ the prescribed bnd. cond. for the fields.

↓ generating f'l of connected corr. f. in the CFT.

  - arbitrary corr. f. can be obtained by differentiating $W_{CFT}[\phi_0(x)]$ w.r.t. $\phi_0(x)$

  - the above dict. is true in general (bulk = string. th.) but we will concentrate on the <u>sugra limit</u> in which $Z_{AdS}[\phi_0] \underset{\text{saddle pct}}{\approx} e^{-\underbrace{S_{sugra}[\phi_0]}_{\text{bulk on-shell action w/ bnd. cond } \phi_0}}$

    $N \to \infty$, λ large

  - both sides are in fact divergent : $Z_{AdS}[IR] \Leftrightarrow Z_{CFT}[UV]$

    $\Rightarrow$ need to renormalize in order to obtain a finite answer holographic renorm.

  - therefore, we <u>first</u> renormalize, then take functional deriv. w.r.t. ϕ_0

  - will show how this works in practice for the 2pf.

(include Petrica lecture notes)

- thus, more concretely, in this case the AdS/CFT dict. can be written as

$$Z_{bulk}\left[\phi(z,x^\mu)\Big|_{z\to 0} = z^{d-\Delta}\phi_0(x)\right] = \left\langle e^{-\int d^dx\, \phi_0(x)\, O_\Delta(x)} \right\rangle_{CFT}$$

generating f. of gauge-invar ops. in the CFT $e^{-W_{CFT}[\phi_0]}$

- arbitrary corr. f. can be obtained by differentiating $W_{CFT}[\phi_0(x)]$ w.r.t. $\phi_0(x)$
- true in general (bulk ~ string th) but we will concentrate on the sugra limit $N\to\infty$, λ large, in which $Z_{bulk}[\phi_0] \underset{\substack{\uparrow \\ \text{saddle pt}}}{\approx} e^{-\underset{\substack{\uparrow \\ \text{bulk on-shell action w/ bnd. cond } \phi_0}}{S_{sugra}[\phi_0]}}$

⑥ Two-point function

- only quadratic part of the action needed $S = \int d^{d+1}x \sqrt{g}\, \frac{1}{2}\left((\partial\phi)^2 + m^2\phi^2\right)$
- the soln is as before $\phi(z,x) = \phi_0\, z^{d-\Delta} + \dots \quad + \phi_\Delta z^\Delta + \dots$
- $\phi_0(x^\mu)$, $\phi_\Delta(x^\mu)$ are indep from the p.d.v of the near bnd. analysis; however, they get related by requiring that the soln not blow up ~~in~~ as $z\to\infty$
- solving the wave eqn in momentum sp $\phi(z,x) = e^{ip_\mu x^\mu} f(z)$, in Eucl AdS

sols $\quad z^{\frac{d}{2}} I_\nu(pz) \quad \& \quad z^{\frac{d}{2}} K_\nu(pz) = z^{\frac{d}{2}}\left(I_\nu(pz) - I_{-\nu}(pz)\right)$

$p = |p| \qquad I_\nu = \left(\frac{pz}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} \qquad = z^{\frac{d}{2}+\nu}\left\{\left(\frac{p}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} - z^{\frac{d}{2}-\nu}\left(\frac{p}{2}\right)^{-\nu}\frac{1}{\Gamma(1-\nu)}\right.$

$\Rightarrow \phi_\Delta = -\left(\frac{p}{2}\right)^{2\nu} \frac{\Gamma(1-\nu)}{\Gamma(\nu+1)}\phi_0$

- let us now evaluate the on-shell action

  - rather $\delta S_{on\text{-}shell}[\phi_0] \leftrightarrow \langle O_\Delta(x^\mu)\rangle_{\phi_0}\,\delta\phi_0$

$$\delta S = \int d^{d+1}x \sqrt{g}\,\underbrace{(m^2\phi - \Box\phi)}_{0}\,\delta\phi + \int d^dx \sqrt{\gamma}\, \underbrace{n^\mu}_{\text{normal to the bnd @ } \underbrace{z=\epsilon}_{\substack{\text{regulator} \\ \text{IR/UV}}}} \partial_\mu\phi\, \delta\phi$$

$-\frac{z}{\ell}\partial_z$

$$\delta S_{on\text{-}sh} = -\int d^dx \; \frac{z}{\ell}\,\partial_z\phi\,\delta\phi\cdot\left(\frac{\ell}{z}\right)^d = -\int_{z=\varepsilon} d^dx \left(\frac{\ell}{z}\right)^{d-1}\Big[\;\phi_0(d-\Delta)\,z^{d-\Delta-1}+\dots$$

$$\dots + \Delta\,\phi_\Delta\, z^{\Delta-1}+\dots\Big]\cdot\Big(\,\delta\phi_0\;z^{d-\Delta}+\dots\;+\delta\phi_\Delta\, z^0+\dots\Big)$$

Note :

- coeff $\phi_0\delta\phi_0$ divergent $\dfrac{\varepsilon^{2d-2\Delta-1}}{\varepsilon^{d-1}} = \varepsilon^{d-2\Delta}$ div for $\Delta > \frac{d}{2}$
- subleading divs $\partial^n\phi_0\,\delta\,\partial^n\phi_0$
- finite term (ε^0) $\quad -\big(\,\phi_0(d-\Delta)\,\delta\phi_\Delta + \Delta\,\phi_\Delta\,\delta\phi_0\big)$

does not have a good variational principle, as only ϕ_0 should be held fixed (so, $\delta S_{onsh} \propto \delta\phi_0$)

- to obtain meaningful & finite results we should - as in QFT - add bnd. counterterms to remove the divs. In AdS/CFT, the counterterms are local, Lorentz invar, & depend only

⑦ intrinsically on the bnd geometry ($\phi|_\varepsilon$, $\gamma_{\mu\nu}|_\varepsilon$ etc.).

→ holographic renormalization

- easy to see the leading divergence can be absorbed via the counterterm $\quad S_{ct} = \frac{c}{\ell}\int d^dx\,\sqrt{\gamma}\;\phi^2(x,z)\Big|_{z=\varepsilon}$ $\quad$ choosing c appropriately

$$\delta S_{ct} = \frac{2c}{\ell}\int_{z=\varepsilon} d^dx \left(\frac{\ell}{z}\right)^d \big(\phi_0\;z^{d-\Delta}+\dots\big)\big(\delta\phi_0\;z^{d-\Delta}+\dots\big)$$

$$-\,\phi_0\,\delta\phi_0\,(d-\Delta) + 2c\,\phi_0\,\delta\phi_0 = 0 \qquad \Rightarrow \quad c = \frac{d-\Delta}{2}$$

- this ct. also modifies the coeff. of the subleading terms (div. or not) & in part, the finite term coeff. becomes

$$-\big(\phi_0(\cancel{d}-\Delta)\,\delta\phi_\Delta + \Delta\,\phi_\Delta\,\delta\phi_0\big) + (d-\Delta)\big(\phi_{\cancel{0}}\,\delta\phi_\Delta + \phi_\Delta\,\delta\phi_0\big)$$

$$= (d-2\Delta)\,\phi_\Delta\,\delta\phi_0 \qquad \text{problematic term cancels!}$$

& $\delta S \propto \delta\phi_0$

- the subleading divs can be cancelled by ct $\int d^dx\,\sqrt{\gamma}\;\phi\,\Box^n\phi$ which do not affect the finite term.

- Thus $\frac{\delta S_{\text{on-shell}}[\phi_0]}{\delta\phi_0} = (2\Delta - d)\,\phi_\Delta = \langle O_\Delta \rangle_{\phi_0}$

$\Rightarrow$ the coeff ϕ_Δ in the near-bnd. expansion of the scalar repr. the expectation value of the dual bnd. operator

- to obtain the 2pt-f $\langle O_\Delta O_\Delta \rangle = \frac{\delta \langle O_\Delta \rangle}{\delta \phi_0} \propto p^{2\nu}$ in momentum space $\langle O(p) O(-p$

- Fourier-transforming $\int d^dp\, e^{ip\cdot x}\, p^{2\nu} \sim \frac{1}{(x_1 - x_2)^{2\Delta}}$ $\Delta = \frac{d}{2}+$

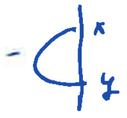
  exactly the 2p.f of an operator of dim. Δ.

Comments

- here, the scalar; higher spin fields very similar $\rightarrow$ higher spin ops in CFT

- for massless gauge fields in the bulk $(A_\mu, g_{\mu\nu}) \rightarrow$ cons. currents on bnd; cons. eqn. obtained from near-bnd-analysis (hologr. Ward identities, including anomalies)

- interesting to note $\frac{1}{|x-y|^{2\Delta}}$ is $e^{-\Delta \ell}$ $\uparrow$ ren. geod. length b/w the pcts x, y on the bnd



  also holds more gen. in non-vac. spt.

Higher-point functions

- need to include interaction terms in the bulk. e.g. for $\frac{\lambda \phi^{n+1}}{n+1}$, the modified e.o.m is

$$(\Box - m^2)\phi = \lambda \phi^n$$

- since λ is small, the sol'n can be expanded perturbatively as

$$\phi = \phi^{[0]} + \lambda \phi^{[1]} + \lambda^2 \phi^{[2]} + \dots$$

where $(\Box - m^2)\phi^{[0]} = 0$ $\qquad (\Box - m^2)\phi^{[1]} = \lambda \phi^n_{[0]}$ $\qquad$ etc.

therefore $\phi^{[1]}(y) = \int d^{d+1}y' \sqrt{g} \underbrace{G_{bb}(y|y')}_{} \phi^n_{[0]}(y')$

bulk-to-bulk propagator $(\Box_y - m^2) G_{bb}(y|y') = \frac{1}{\sqrt{g}}\delta^{d}(y$

- the zeroth order sol'n $\phi_{[0]}$ is itself det. by the bnd. value ϕ_0, of the field using the bulk-to-bnd propagator $K_{b\partial}(y|x)$

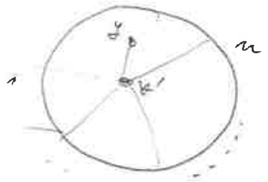
$$(\Box_y - m^2) K_{b\partial}(y|x) = 0 \qquad \lim_{z\to 0} K_{b\partial}(z, x'|x) \propto z^{d-\Delta} \delta^{(d)}(x-x')$$

$$\phi_{[0]}(y) = \int d^d x \; K_{b\partial}(y|x)\, \phi_0(x)$$

soln unique euclidean

the sol'n turns out to be (Poincaré coord) $K_{b\partial} = \left(\frac{z}{z^2 + (x-x')^2}\right)^\Delta$

- thus, the perturbative bulk field can be constr. via a Witten diagram

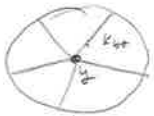


- the answer for the $\overset{n+1}{\vee}$ corr. f^n is given by extracting the coeff. of the z^Δ term from $\phi^{[1]}$ (= 1p.f in presence of the source) & differentiating n times

- since $\lim_{z\to 0} G_{bb}(z, x | z', x') = \frac{z^\Delta}{2\nu} K(z|z', x')$, the final answer for the nex-pf is

- the end result for the $n+1$ - point function is

$$\langle O(x_1)\dots \quad O(x_n)\rangle = -\frac{\delta S^{ren}_{cls}}{\delta\phi_{(0)}(x_1)\dots\delta\phi_{(0)}(x^n)} = \lambda\int d^{d+1}y \; K_{b\partial}(x_1,y)\dots \quad K_{b\partial}(x_{n+1},y)$$



- more generally, CFT corr. f. can be computed via AdS diag. w/ pts ending on the bnd.

- ∃ an alternate way of computing CFT corr. as bnd limits of bulk ones.

$$\langle O_1(x_1)\dots \quad O_n(x_n)\rangle_{CFT} = \lim_{z_i\to 0}\prod_i 2\nu_i z_i^{-\Delta_i}\langle\phi_1(z_1,x_1)\dots \quad \phi_n(z_n,x_n)\rangle$$

Comments, continued

- holographic corr. f. are computed using diagrammatic rules (Witten diagrams) that are analogous to Feynman diagrams

- the holographic dict. we discussed consisted of coupling the bnd. CFT to sources (identified w/ the non-normalizable modes of bulk fields) & differentiating w.r.t. them to obtain corr. f. (GKPW) An alternative way of computing correlation f. is the extrapolate dictionary . whereby bnd. holographic correlators are obtained as bnd. limits of bulk ones

$$\langle O_1(x_1) \dots O_n(x_n) \rangle_{CFT} = \lim_{z \to 0} z^{-\sum_i \Delta_i} \langle \phi_1(x_1, z) \dots \phi_n(x_n, z) \rangle_{bulk}$$

which is conjectured to be equivalent (true, but careful ren. ops.)

  - follows from the fact that $\lim_{z' \to 0} G_{bb}(x, z; x', z') = z'^{\Delta} \underset{b\partial}{K_\Delta}(x, z; x')$

$$G_{bb}(y, y') = e^{-\Delta \sigma(y, y')} {}_2F_1 (\Delta, \tfrac{d}{2}; \Delta + 1 - \tfrac{d}{2}; e^{-2\sigma}) \qquad e^{\sigma} = \frac{1 + \sqrt{1 - \xi^2}}{\xi} \qquad \xi = \frac{2zz'}{z^2 + z'^2 + |x - x'|^2}$$

geodesic dist.

  - however, in general, these 2 dictionaries are not equivalent!

- above, we computed holographic 2pf → perfect match to CFT 2p.f. (once spectrum fixed)
- same for 3pf. (home work) ⟵ ↗ purely kinematical

- starting from 4 p.f., the functional form of the corr. f. depends on the coupling $\sum f(u, v)$ cross ratios
  - features related to local bulk physics only emerge @ strong coupling (bulk pt. singularity). expected to be smoothened out by finite α' effects