Introduction to holography - Lecture XII

- brief review of AdS/CFT aspects we discussed so far

– derivation of the corresp. b/w type IIB string th. on $AdS_5 \times S^5$ and $\mathcal{N}=4$ $SU(N)$ SYM in $d=4$

& matching of parameters

$\mathcal{N}=4$ SYM	IIB on $AdS_5 \times S^5$	
g_{YM}^2	g_s	modulus
$\lambda = g_{YM}^2 N$	ℓ_{AdS}^4 / α'^2	
N	$\left(\frac{\ell_{AdS}}{\ell_P}\right)^4$	, or $G_{10} \sim \frac{1}{N^2}$

$\Rightarrow$ $\frac{1}{N}$ corr. in CFT $\leftrightarrow$ Q.Grav. corr.

$\frac{1}{\sqrt{\lambda}}$ corr. in CFT $\leftrightarrow$ α' corr.

– when perturbatively expanding the type IIB action around the $AdS_5 \times S^5$ backgnd, interaction terms are suppressed by $1/N$ & $1/\sqrt{\lambda}$ $\Rightarrow$ in the large N limit, fields are approx. free (good starting pt. for the $1/N$ expansion).

– this analysis/param. holds for the specific duality b/w IIB/$AdS_5 \times S^5$ & $\mathcal{N}=4$ SYM, but very similar statements hold for other examples (e.g. for IIB/$AdS_3 \times S^3 \times T^4 \times \mathbb{Z}$ a 2d CFT known as the D1-D5 CFT, the role of N is played by the CFT central charge, $c = 6 Q_1 Q_5$. The coupling g_{YM} (& θ) get gen. to an 84-dim'l coupling space) explain.

- another important aspect we discussed is the UV/IR connection

$$ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right) \qquad z \approx \text{energy scale}$$

$$E_{CFT} \sim \frac{\ell}{z} E_{local}(z) \qquad |\Delta x|_{CFT} = \frac{z}{\ell} |\Delta x|_{local}$$

$\Rightarrow$ large distances in AdS ($z \approx 0$) $\longleftrightarrow$ short dist/high energies in CFT

IR $\qquad\qquad$ UV

– IR cutoff in AdS ($z = \varepsilon$) $\leftrightarrow$ UV cutoff in CFT

- an obvious first check of the proposed duality is the matching of symmetries

  - symmetries of the CFT vacuum state $\leftrightarrow$ isometries of AdS ($\times S^{?}$...)

$\mathcal{N}=4$ SYM	IIB / $AdS_5 \times S^5$
conf. gp $SO(4,2)$	isometries AdS_5
global internal $SO(6)$	isometries S^5
(32) fermionic sym	same

D1-D5 CFT	IIB / $AdS_3 \times S^3 \times T^4$
conf. gp $SO(2,2)$	isometries AdS_3
global int. $SO(4)$ $\times U(1)^4$	isometries S^3 & T^4
susy	✓

  - symmetries of the theory $\leftrightarrow$ asymptotic symm of the dual grav. th.

    - look @ a AdS spt w/ appropriate bnd cond for metric & other fields
    - allowed gauge transf. are those diffeos / gauge transf. in the bulk that preserve the bnd. cond.
    - from there, need to discard the trivial diffeos / gauge transf that fall off too fast to contribute to $\forall$ phys cons. charges
    - the quotient $\frac{\text{allowed}}{\text{trivial}}$ = asymptotic symm (large gauge transf.)

  - e.g. asympt. symm. diffeos $AdS_5 \times S^5 \to \underbrace{SO(4,2)}_{\text{diffeos on AdS}} \times \underbrace{SO(6)}_{\text{on } S^5}$ & gauge symm. from AdS_5 pers.

    – // – $AdS_3 \times S^3 \to Vir_L \times Vir_R \times \widetilde{SU(2)}^{KM}_L \times \widetilde{SU(2)}^{KM}_R$

  - in general gauge "symm" in the bulk $\leftrightarrow$ global symm in bnd.

    gauge fields in the bulk $\leftrightarrow$ global cons. currents in the bnd

  - we saw explicitly in the last lecture how the gauge red. of Maxwell on AdS $\Rightarrow$ global phase rot on ∂AdS.

- one also has a match of the Hilbert sp & of the spectrum of perturbations of AdS ($\times S$) w/ the CFT one

- + symmetries ; reps. of the isometry gp $SO(d,2)$ $\leftrightarrow$ reps of the conformal gp. $SO(d,2)$
  (bulk) (bnd.)

- remember that (e.g., when acting on bulk scalars) the quadratic Casimir $\frac{1}{2}J_{AB}J^{AB}$ $\leftrightarrow$ AdS Laplacian

- for a free scalar $(\Box_{AdS} - m^2)\phi = 0 \Rightarrow \frac{1}{2}J_{AB}J^{AB}\phi = m^2 \ell^2_{AdS}\phi$
  - eigenval conf. Casimir $\Delta(\Delta - d) = m^2\ell^2$

  $\Rightarrow$ free scalar field in AdS $\subset$ conformal family of a primary op. of dim $\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2}$

- energies of fields in global AdS_{d+1} $\leftrightarrow$ energies of the CFT_d on the cylinder $S^{d-1}\times\mathbb{R}$
  $\leftrightarrow$ conformal dims. of operators on the plane. (holds also for heavy)

  ($\Rightarrow$ spectrum of conf. dims. of the CFT $\Leftrightarrow$ energy spectrum in gl. AdS)

- CFT spectrum : conformal families $\Delta + n \in \mathbb{N}$
  (Δ: primary)

- the dual to a CFT primary state $\leftrightarrow$ primary wavef. that is annihilated by the special conf. gen K_μ in AdS., & has eigenval. Δ for the dilatation (gl. eng.)

- descendant states are built by acting w/ P_μ. $\Rightarrow$ integer-spaced spectrum

- let us show that the spectrum of (scalar, for simplicity) excitations in global AdS is discrete : the sols to the wave eqn. are

$$\Phi = e^{-i\omega t}\,\chi(\theta)\,Y^{S^{d-1}}_{\ell\{m\}} \qquad ds^2_{gl\,AdS} = \frac{\ell^2}{\cos^2\theta}\left(-d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega^2_{S^{d-1}}\right)$$

Y: spherical harm. on S^{d-1} $\qquad \Box_{S^{d-1}} Y_{\ell\{m\}} = -\ell(\ell+d-2)\,Y_{\ell\{m\}}$

- the sol^n_s to the wave eqn. are $\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 \ell^2_{AdS}}$

$$\chi^{\pm}(\theta) = (\cos\theta)^{\Delta\pm} (\sin\theta)^{\ell}\, {}_2F_1\left(\frac{\Delta_\pm + \ell + \omega}{2}, \frac{\Delta_\pm + \ell - \omega}{2}, \Delta_\pm + 1 - \frac{d}{2}, \cos^2\theta\right)$$

- for $m^2_{(>0)}$ large enough, only the Δ_+ soln is normalizable @ the AdS bnd. $(\theta = \frac{\pi}{2})$

- also want sol^n smooth in the interior, near $\theta \simeq 0$. Using the id

$${}_2F_1(a,b,c,z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\, {}_2F_1(a,b,a+b+1-c,1-z) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} \cdot {}_2F_1(c-a, c-b, 1+c-a-b, 1-z)$$

$$c-a-b = \Delta + 1 - \frac{d}{2} - \Delta - \ell = 1 - \frac{d}{2} - \ell < 0 \Rightarrow (1-z)^{\#} \text{ divergent!}$$

$\Rightarrow \Gamma(b)$ must have a pole $\Rightarrow$ $\omega = \Delta + \ell + 2n$

integer-spaced, as in CFT.

- works similarly for fields w/ spin

boundary theory	bulk theory
reps. conformal gp. $SO(d,2)$	reps isometry gp. $SO(d,2)$
local single trace operators (Δ family, w $\mathcal{C}_2 = \Delta(\Delta-d)$)	bulk fields $\mathcal{C}_2 = m^2\ell^2$ (scalar)
vector ops J_μ conserved –//–	vector field A_M gauge field
tensor ops $T_{\mu\nu}$ stress tensor	tensor fields h_{MN} graviton

KK red. on S^5

- free approx is a good one @ large N $\rightarrow$ large N factorization

GFF on bnd $\leftrightarrow$ free field bulk.

- to test AdS/CFT, should check that the spectrum matches on both sides; however weak/strong coupling duality (hard to compute one side when the other is weakly-coup.)
  - can look for dimensions protected by supersymmetry ($\frac{1}{2}$ BPS) - do not dep. on λ.
  - use integrability techniques.

- e.g. can match tower of KK modes of 10d sugra reduced on S^5 - fit into short reps. of susy alg (protected) can be matched to corresp. CFT towers e.g. $tr(\phi^{I_1} .. \phi^{I_n})$ symm. traceless w/ $\Delta = n$. $\rightarrow$ corresp. $m^2 \ell^2 = n(n-4)$ + other towers.
  single-trace
  multi-tr $\Leftrightarrow$ multi-part. states

- string modes have $m^2 \ell_{AdS}^2 \approx \frac{\ell_{AdS}^2}{\alpha'} \sim \sqrt{\lambda} \gg 1$ in the sugra limit $\Rightarrow$ operators whose dims. scale as $\Delta \sim \lambda^{1/4}$ (does not. scale w/ N). These corresp. to unprotected operators in $\mathcal{N}=4$ SYM that acquire a large anomalous dim. & prediction

- finally, since the bulk is a theory of gravity, one can have black holes. Their energies $E \propto N^2$ & entropy $S_{BH} \sim N^2$

E $\uparrow$ $N \to \infty$

N^2 Black holes $S \propto N^2$ (in $\mathcal{N}=4$ SYM).

$\lambda^{1/4}$ strings extends

$O(1)$ Fock space of weakly-int. particles (KK modes from 10d, $s \leq 2$) (also includes multi traces) ops $Tr(\phi ..) Tr(\phi ..) \Leftrightarrow$ multiparticle states in AdS