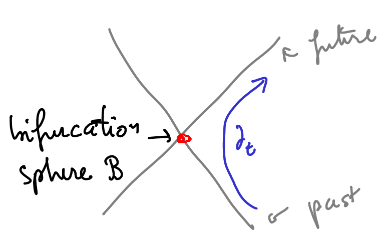
The four laws of black hole mechanics

- we saw black holes generically form as a result of gravitational collapse (assuming cosmic censorship)
- due to the uniqueness theorems, these final states are characterised by a small # of parameters (M, J), just as a thermodynamic system is
- we will now further explore the analogy between b.h. & thermodynamics
- a useful (though not very detailed) review w/ refs to the original literature is: R. Wald: "The thermodyn. of b.h" in Living reviews in relativity
- see also Ted Jacobson: "Introductory notes on b.h. thermodynamics"

Setup: consider a stationary black hole space-time. It has been shown that in vacuum or electrovac GR, the horizon must be a Killing horizon, i.e. a null hypersurface to which a Killing vector field is normal (and thus $\xi^2 = 0$ on $\mathcal{H}$)

- as long as the b.h.s are non-extremal & we look @ the maximally analytically extended spt, these horizons are bifurcate Killing horizons



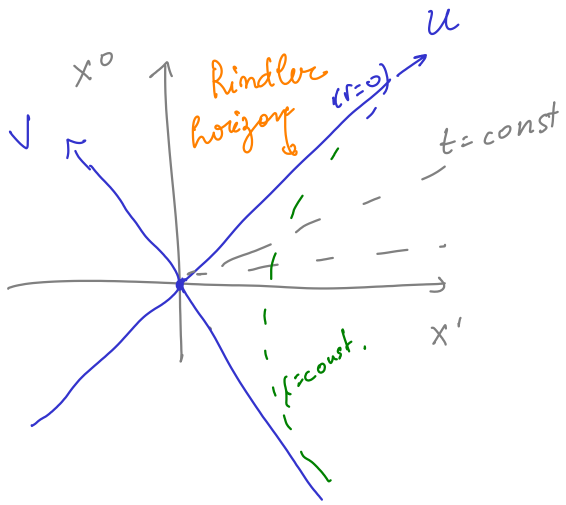
- pair of orthogonal null surfaces = Killing horizons w.r.t the same K.V.
- the Killing vect. ξ^μ vanishes on $\mathcal{B}$

Example : Rindler space

- patch of (2d) Minkowski in coord. adapted to the experience of an accelerated observer

$$ds^2 = -r^2 dt^2 + dr^2 + d\vec{x}_\perp^{\,2}, \quad \text{obtained via} \quad \begin{cases} x^0 = r \sinh t \\ x^1 = r \cosh t > 0 \end{cases}$$

↑ patch $\mathbb{R}^{1,1} \times \mathbb{R}^{d-2}$



- $\xi = \frac{\partial}{\partial t}$ is a K.V. $\rightarrow$ boost in Minkowski

$$\partial_t = \frac{\partial x^0}{\partial t}\partial_{x^0} + \frac{\partial x^1}{\partial t}\partial_{x^1} = \left(x^1 \partial_0 + x^0 \partial_1\right)$$

- the lines $x^1 = \pm x^0$ form a bifurcate Killing horizon, which has 2 components $\cap$ $\underbrace{x^0 = x^1 = 0}_{\mathcal{B}}$ w/ $\xi|_{\mathcal{B}} = 0$

- check I : • $\xi^2 = x_0^2 - x_1^2 = 0$ for $x^1 = \pm x^0$

- in Rindler coord $x^0 = \pm x^1$ is obtained via $r \to 0$, $t \to \pm\infty$ w/ $re^{\mp t}$ fixed

- more precisely, by intro. $u, V = x^0 \pm x^1 = \pm r e^{\pm t}$, the horizon's components are at $V = 0$, u fixed & $u = 0$, V fixed < 0

- in terms of these $\xi = u\partial_u - V\partial_v$

check II

- $\xi \perp$ horizon $\Rightarrow \xi_\mu \propto n_\mu \propto \partial_r = \frac{\partial u}{\partial r}\partial_u + \frac{\partial V}{\partial r}\partial_v = e^t \partial_u - e^{-t}\partial_v \begin{cases} \propto \partial_u , & t > 0 \\ \propto -\partial_v , & t < 0 \end{cases}$

Other examples : $r = 2M$ in Schwarzschild, for ∂_t

$r = r_+$ in Kerr, for $\partial_t + \Omega_H \partial_\varphi$ w/ Ω_H = angular velocity of the horizon

Some properties of Killing horizons : (see e.g Townsend 9707012)

- let ℓ^μ be the normal to the null hypersurface $\mathcal{N}$
- ℓ^μ null $\Rightarrow$ ℓ^μ is also a tg. vector in $\mathcal{N}$, i.e. $\ell^\mu = \frac{dx^\mu}{d\lambda}$ for some curve $x^\mu(\lambda)$ in $\mathcal{N}$
- the curves $x^\mu(\lambda)$ are geodesics : $\ell^\lambda \nabla_\lambda \ell^\mu \propto \ell^\mu$ (see e.g. Townsend)

generators of $\mathcal{N}$

$\ell_\mu \perp \Rightarrow \ell_\mu = f \partial_\mu S$ ← function that specifies hypersurf.

$$\Rightarrow \ell^\lambda \nabla_\lambda \ell_\mu = \ell^\lambda \nabla_\lambda (f \partial_\mu S) = \underbrace{\ell^\lambda \partial_\lambda f \frac{\ell_\mu}{f}} + \ell^\lambda f \nabla_\lambda \partial_\mu S = \ell^\lambda \partial_\lambda \ln f \, \ell_\mu +$$

$$+ \ell^\lambda f \nabla_\mu \partial_\lambda S = \cancel{\#} \, \ell_\mu = \ell^\lambda f \nabla_\mu \ell_\lambda/f = \underbrace{\ell^2}_{0 \text{ on } \mathcal{N}} \frac{\partial_\mu f}{f} + \underbrace{\frac{1}{2} \nabla_\mu \ell^2}_{\perp \forall t^\mu = tg.\text{ vect to } \mathcal{N}}$$

$\Rightarrow \propto \ell^\mu$ q.e.d.

- affine param: $\ell^\lambda \nabla_\lambda \ell^\mu = 0$
- since $\xi^\mu \propto \ell^\mu$, the Killing vect $\perp \mathcal{N}$ satisfies

surface gravity

$$\boxed{\xi^\mu \nabla_\mu \xi^\nu = \kappa \, \xi^\nu}$$

geodesic eqn. in non-affine param

ξ has a natural normaliz. @ ∞

- this leads to the 1st interpretation of surface gravity :

since $\xi^\mu = f \ell^\mu$ (affine param), we have $\xi^\mu \nabla_\mu (f \ell^\nu) = \underbrace{\xi^\mu \partial_\mu \ln f}_{\kappa} \, \xi^\nu$

so κ measures the failure of the Killing param, v : $\xi^\mu \partial_\mu v = 1$, to equal λ ← affine param.

can show $f = \lambda \kappa$

$\partial_v \ln f = \kappa \Rightarrow f = f_0 e^{\kappa v}$, $\partial_v = f_0 e^{\kappa v} \partial_\lambda \Rightarrow \lambda = \frac{f_0}{\kappa} e^{\kappa v} = f/\kappa$

· for the 2nd interpretation note that · proper acceleration particle along ξ^μ is $a^\mu = u^\lambda \nabla_\lambda u^\mu$, w/ $u^\mu = \frac{\xi^\mu}{\sqrt{-\xi^2}}$

$$\Rightarrow a^\mu = \frac{\xi^\lambda}{\sqrt{-\xi^2}} \nabla_\lambda \frac{\xi^\mu}{\sqrt{-\xi^2}} = \frac{\xi^\lambda \nabla_\lambda \xi^\mu}{-\xi^2} + \underbrace{\xi^\lambda \xi^\rho \xi^\mu \nabla_\lambda \xi_\rho}_{0}$$

Noting that
$$\kappa^2 = \lim_{\to N} \frac{(\xi^\mu \nabla_\mu \xi_\lambda)(\xi^\nu \nabla_\nu \xi^\lambda)}{\xi^2}$$
acceleration orbits ξ × ξ^2 redshift

so $\kappa = \lim_{\to \text{hor}} (a V)$ = force that must be exerted @ infinity to hold a unit test mass in place, hence the name

V: redshift fact $\sqrt{-\xi^2}$

· also possible to show that $\kappa^2 = -\frac{1}{2} \nabla^\mu \xi^\nu \nabla_\mu \xi_\nu \big|_N$ (see e.g. Townsend) Wald p. 332

· 3rd interpretation: κ also given by 2π/periodicity of the euclidean time coord. chosen to ensure that Euclidean b.h. geom. is smooth (no conical defect)

Example : Rindler space

can also be obt. from $\perp$ + affine parameter.

- $\xi^\mu = \partial_t$, while null geod $\parallel$ future horizon is ∂_U , $\xi \simeq U\partial_U$ near future hor
- $\Rightarrow f = U$ & $\kappa = \xi^\lambda \partial_\lambda \ln f = U\partial_U \ln U = 1$

- also from periodicity : $dr^2 + r^2 dt_E^2 \Rightarrow t_E \sim t_E + 2\pi \Rightarrow \kappa = 1$

- otherwise $\xi^\lambda \nabla_\lambda \xi_\mu = -\xi^\lambda \nabla_\mu \xi_\lambda = -\frac{1}{2}\partial_\mu \xi^2 \underset{-r^2 = UV}{\sim} -\frac{1}{2}\left(V\delta_\mu^U + U\delta_\mu^V\right) = \pm \xi_\mu \Big|_{\mathcal{N}} \Rightarrow \kappa = \pm 1$

near hor. $\to 0$

Exercise : 1 Compute the surface gravity associated w/ the Killing horizon of the Schwarzschild black hole

2. Compare the result to $2\pi/$ the periodicity of the Euclidean Schwarzschild time necessary to have a smooth geom.

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega_2^2 \qquad t = -i t_E$$

near $r = 2M + \rho^2$

$$dt_E^2\left(1 - \frac{2M}{2M+\rho^2}\right) + \frac{(2\rho\, d\rho)^2}{\frac{2M+\rho^2-2M}{2M+\rho^2}} = \frac{\rho^2 dt_E^2}{2M} + 8M d\rho^2 \simeq 8M\left(d\rho^2 + \frac{\rho^2 dt^2}{16M^2}\right)$$

$\Rightarrow$ no conical singularity if $t_E \sim t_E + 2\pi \cdot 4M \Rightarrow \boxed{\kappa = \frac{1}{4M}} \times G$

Zeroth law of b.h. mechanics

The surface gravity, κ, of a stationary b.h. is constant over the event horizon.

Pf: 2 steps: 1^{st}, prove that $\kappa = \text{const.}$ on orbits of ξ, $(\xi^\mu \partial_\mu \kappa^2 = 0)$

(Hint: use K.V. id $\nabla_\rho \nabla_\mu \xi^\nu = R^\nu{}_{\mu\rho\sigma} \xi^\sigma$)

2^{nd}, prove that $\kappa = \text{const}$ on $\mathcal{B}$ $\left(t^\lambda \nabla_\lambda \kappa^2 = -\nabla^\mu \xi^\nu t^\lambda R_{\nu\mu\lambda\sigma} \xi^\sigma \big|_{\mathcal{N}} \right)$

$\forall t^\lambda$ tg to $\mathcal{N} \to \mathcal{B}$

since $\kappa = \text{const}$ on orbits of ξ, $\kappa = \text{const}$ on $\mathcal{N}$ if it is const on $\mathcal{B}$

The reason this is called the zeroth law is that, as we will show, κ is analogous to a temperature, so this is reminiscent of the zeroth law of thermodyn: temperature is uniform everywhere in a system in thermal equilibrium.

- note our proof is purely geometrical, but makes use of the bifurcate structure of the Killing horizon. $\exists$ a different proof which does not make use of it, but uses instead Einstein's eqns + dominant energy cond. (see e.g. Wald's book)

The first law

$$\frac{\kappa}{2\pi} \frac{\delta A}{4G} = \delta M - \Omega_H \, \delta J \quad (+ \ldots)$$

- several different ways to show it

1. Check nearby stationary solutions satisfy it

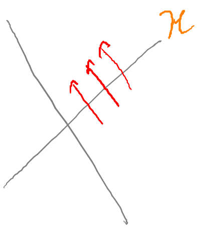
Example: Schwarzschild

$R_s = 2GM \Rightarrow A = 16\pi G^2 M^2$, $\kappa = \frac{1}{4GM}$ $\quad \frac{1}{8\pi GM} \frac{\delta(16\pi G^2 M^2)}{4G} = \delta M$

- note that since b.h. absorb energy but do not emit (classically) $\delta M > 0 \Rightarrow \delta A > 0$

② Physical process version of the 1st law

Consider a quasistatic process in which mass is added to the black hole. We thus start w/ a stationary black hole & alter it by some infinitesimal physical process. We assume the b.h. then settles down to a new stationary final state.



- $\Delta T_{\mu\nu}$ = stress energy of matter thrown in. The change in the mass & angular momentum of the b.h. are

$$\Delta M = \int_0^\infty \underbrace{d\lambda}_{\text{affine param on } \mathcal{H}} \int d^2\Omega \; \Delta T_{\mu\nu} \underbrace{\tau^\mu}_{\partial_t} \underbrace{k^\nu}_{\perp \mathcal{H}} \qquad \Delta J = -\int_0^\infty d\lambda \int d^2\Omega \; \Delta T_{\mu\nu} \underbrace{\varphi^\mu}_{\partial_\varphi} k^\nu$$

- the change in the area of the horizon is $\Delta A = \int d^2\Omega \, d\lambda \, \theta(\lambda)$

- a Killing horizon has const. cross-sectional area, so $\theta = 0 \Big|_{\text{initial}}^{\text{final}}$ (in fact, $B_{\mu\nu} = 0$ for a Killing horizon)

uses Einstein's equs — see e.g. Townsend

- then, by Raychaudhuri's eqn, $\dfrac{d\theta}{d\lambda} = -8\pi G \, \Delta T_{\mu\nu} k^\mu k^\nu$ (drop θ^2 terms)

$$\Rightarrow \Delta A = \int_0^\infty d\lambda \int_{S^2} d^2\Omega \, \theta = \int_{S^2} d^2\Omega \, \underbrace{\lambda\theta \Big|_0^\infty}_{0 \text{ b/c } \theta_{i,f} = 0} - \int_0^\infty d\lambda \cdot \int d^2\Omega \, \lambda \frac{d\theta}{d\lambda} =$$

$$= 8\pi G \int_0^\infty d\lambda \, d^2\Omega \, \underbrace{\lambda \, \Delta T_{\mu\nu} k^\mu}_{\frac{1}{k}\xi^\mu} k^\nu = \frac{8\pi G}{k} \left(\Delta M - \Omega_{\mathcal{H}} \Delta J \right)$$

since $\xi^\mu = \tau^\mu + \Omega_{\mathcal{H}} \varphi^\mu$

$$\Rightarrow \frac{k \delta A}{8\pi G} = \delta M - \Omega_H \Delta J$$

- looks like "1st law" of thermodyn $T dS = dM + \ldots \Rightarrow$ $A_H \propto$ entropy, $k \propto$ temperature

- same argument + the null energy condition, can be used to show that $\Delta A > 0$ $\simeq$ 2^{nd} law of thermodyn.

<u>The third law</u> : it is impossible, by any procedure, no matter how idealized, to reduce κ (surface gravity) to zero by a finite sequence of operations.

(clearly, cannot require that $S \to 0$ as $T \to 0$, violated by extremal b.h.s)

- very hard to bring to extremality by adding charge / spin.

- counterexample recently constr. by matematicians (finely-tuned collapse $\Rightarrow$ exactly extremal Reissner-Nordström in finite time)

- relevance unclear due to large quantum fluctuations for very-near-extremal black holes.