Renormalization of complex scalar field theory

Consider a 4D renormalizable QFT of a complex scalar field ϕ with Lagrangian (here on, when we write a Lagrangian, we mean the renormalized Lagrangian for some choice of the renormalization scale μ)

$$\mathcal{L} = \partial\phi^*\partial\phi - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2 \tag{1}$$

1) Discuss renormalization, paying attention to the global symmetry satisfied by this lagrangian. Derive the full set of RG equations at 1-loop.

2) Consider adding to the above lagrangian a small perturbation of the form

$$\Delta\mathcal{L} = \frac{\eta_4}{4!}\phi^4 + \frac{\eta_4^*}{4!}(\phi^*)^4 \tag{2}$$

with η_4 a complex coupling.

- Discuss the symmetries and, taking them into account, discuss what changes in the renormalization? Which new counterterms, if any, are needed?
- Compute the full set of 1-loop RG equations
- Compute the anomalous dimension of ϕ at the lowest order.

3) Consider now adding to eq. (1) a more elaborated perturbation

$$\Delta\mathcal{L} = \left[\frac{\eta_4}{4!}\phi^4 + \frac{\eta_4^*}{4!}(\phi^*)^4\right] + \left[\frac{\eta_3}{3!}\phi^3 + \frac{\eta_3^*}{3!}(\phi^*)^3\right] \tag{3}$$

- Discuss symmetry and renormalization in general. Which counterms are needed?
- Discuss renormalization at 1-loop and derive the RG equations.
- Discuss the renormalization of terms quadratic in the field (mass terms) up to 2-loops and focuss possibly on those that are structurally new with respect to the case of eq. (1).

4) [Bonus] Consider adding to eq. (1) a perturbation

$$\Delta\mathcal{L} = \frac{\eta_n}{n!}\phi^n + \frac{\eta_n^*}{n!}(\phi^*)^n \tag{4}$$

with some $n > 4$.

- What changes in the renormalization of this QFT? What about the symmetries?
- Compute the 1-loop RG equations to linear order in η_n. What happens when $n \to \infty$ with λ fixed?