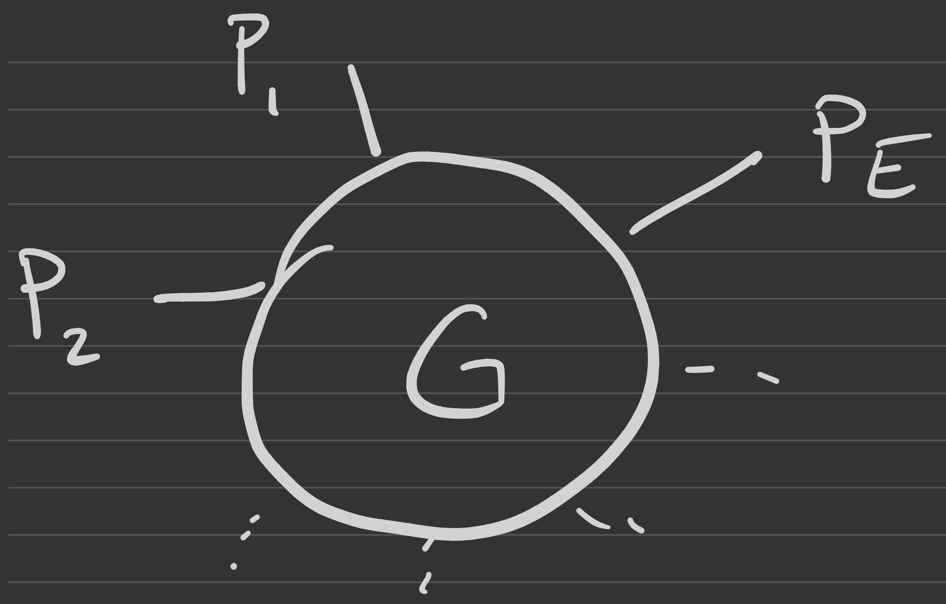
Reading in Collins

- 5.1 — 5.4
- 5.5.1 (5.5.2 if you want details)
- 5.8



E = # external legs
V = # vertices
I = # internal legs
L = # loops

$$\sum d_\alpha \equiv d_v$$

$$G_L = \int \prod_{l=1}^{L} d^D k_l \prod_{e=1}^{I} \Delta(p_e) \prod_{\alpha=1}^{V} f_\alpha(p_{e_\alpha})$$

$$\sim \int K^{LD} \frac{dK}{K} K^{-2I} K^{d_v} = \int K^{\delta(G)} \frac{dK}{K}$$

$$\boxed{\delta(G_L) = LD - 2I + d_v}$$

Renormalized ⇒

$$\boxed{\begin{array}{l} G_L \sim p^{\delta(G_L)} \underset{\displaystyle \downarrow}{F_0(\log p)} + p^{\delta(G_L)-2} F_1(\log p) + \cdots \\ \qquad (\log p)^L + (\log p)^{L-1} + \cdots \qquad \text{by } \underline{\text{Weinberg Th}} \end{array}}$$

△ Inductive "proof" of counterterm locality

- Schematically

$$G_{L+1} = \int d\mu_1 \cdots d\mu_{L+1} \underbrace{I(p, k_1, \dots k_{L+1})}_{\text{all c.t. up to } L\text{-loops}}$$

- can consider $L+1$ distinct integration regions: $R_1, \dots R_{L+1}$

$$R_i = \min\left(|k_1|, \dots |k_{L+1}|\right) = |k_i|$$

$$G_{L+1} = \sum_{i=1}^{L+1} \int d\mu_i \underbrace{\int d\mu_1 \cdots d\mu_{i-1}\, d\mu_{i+1} \cdots d\mu_{L+1}\, I(p, k_1, \dots k_{L+1})}_{G_L^i(p, k_i)\, V^i(p, k_i)\, \Delta^i(p, k_i)}$$

$K_i < K_j \Rightarrow G_L$

smooth for $p \to 0$

- $G_L^i(p, K_i)$ = finite & analytic at $p=0$, $K_i \neq 0$

$L_i \equiv \log \frac{K_i}{\mu}$

for $K_i \gg p$: $\displaystyle\int \frac{dK_i}{K_i} K_i^{\delta_{G_{L+1}}} \left(F_0(L_i) + \frac{p}{K_i} F_1(L_i) + \frac{p^2}{K_i^2} F_2(L_i) \ldots \right)$

$$\Rightarrow \left(\frac{\partial}{\partial p}\right)^{\delta_{G_{L+1}}+1} G_{L+1} = \text{UV-finite}$$

$\Rightarrow$

- $\delta(G_{L+1}) < 0 \quad \Longrightarrow$ finite

- $\delta(G_{L+1}) > 0 \quad \Longrightarrow$

$$\underbrace{\text{divergence} \equiv \text{degree } \delta(G_{L+1}) \text{ polynomial}}_{\substack{\text{renormalized by adding a} \\ \text{local counterterm}}}$$

⚠ Structure of needed counter-terms specified by the $\delta(G)$ of the various n-point functions

$$\cdot\ \delta(G) = D - \overbrace{\left(\frac{D-2}{2}\right)}^{\Delta_\phi} E + \sum_\alpha \overbrace{\left(\frac{D-2}{2}\, n_\alpha + d_\alpha - D\right)}^{-\Delta_\alpha}$$

$$\boxed{\Delta_\alpha = D - d_\alpha - \frac{D-2}{2}\, n_\alpha} \qquad \Downarrow \quad g_\alpha\, \partial^{d_\alpha} \phi^{n_\alpha}$$

⚠ Crucial distinction: $\boxed{\Delta_\alpha \geq 0}$ versus $\boxed{\Delta_\alpha < 0}$

① $\Delta_\alpha \geq 0$: • $\delta(G) \geq 0 \implies E \leq \frac{2D}{D-2}$

$D=4$: ≤ 4

$D=6$: ≤ 3

$D=3$: ≤ 6

$$\text{C.T} \sim \partial^{\delta(G)} \phi^{E} \quad \text{with } \delta(G) + \frac{D-2}{2} E \leq D$$

Ex $\quad \mathcal{L} = (\partial\phi)^2 + m^2\phi^2 + g\phi^3 \quad$ in $D=6$

$$E \leq \frac{12}{4} = 3$$

possible c.T. $\quad \phi^3, \phi^2, \phi, (\partial\phi)^2$

- $\Delta_\alpha \geq 0 \implies$ finite class of counterterms

$\equiv$ Renormalizable

② $\Delta_\alpha < 0$ for some vertex

$$\delta(G) = D - \frac{D-2}{2}E - \sum_\alpha \Delta_\alpha$$

- $\forall E$ $\delta(G)$ can be made > 0 by sufficient insertions of vertex with $\Delta_\alpha < 0$

C.T. $\partial^{\delta(G)} \phi^{E}$ $\qquad \delta(G) + \frac{D-2}{2}E \overset{= D - \sum_\alpha \Delta_\alpha \;\downarrow}{=} \underline{\text{unbounded}}$

$\Rightarrow$ class of necessary counter-terms is infinite

<u>Non-Renormalizable</u>

⚠ Modern perspective

Non-Renormalizable QFT $\equiv$ Effective QFT

- notice $\mathcal{L} \sim g_* \, \partial^d \, \phi^E$

$$[g_*] = -d - \frac{D-2}{2}E + D < 0$$

$$g_* = \frac{1}{M^{d+\frac{D-2}{2}E-D}} \equiv M^{-\Delta_\alpha}$$

At energy E what is the size of effects of g_*?

$$\bar{g}_* = \left(\frac{E}{M}\right)^{d+\frac{D-2}{2}E-D}$$

◉ Loop counting

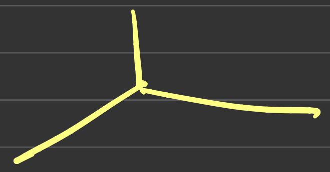
Ex

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{g}{3!}\phi^3$$

$$\phi \longrightarrow \frac{\phi}{g} \implies \mathcal{L} \longrightarrow \frac{1}{g^2}\left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{3!}\phi^3\right]$$

$$\boxed{g^2 \sim \hbar}$$

$\propto g^2$



$\propto 1/g^2$

$$\sim (g^2)^{I-V} = (g^2)^{L-1}$$

$$\sqrt{\hbar} \sim g$$

Ex

$$\lambda_4 \sim \hbar$$

$$\frac{1}{2}(\partial\phi)^2 + \frac{\lambda_4}{4!}\phi^4 \xrightarrow{\phi \to \frac{1}{\sqrt{\lambda}}\phi} \frac{1}{\lambda_4}\left[\frac{1}{2}(\partial\phi)^2 + \frac{\phi^4}{4!}\right]$$

$$\lambda_4^{I-V} = \lambda_4^{L-1} = (g_4^2)^{L-1}$$

$$\lambda_4 \equiv g_4^2$$

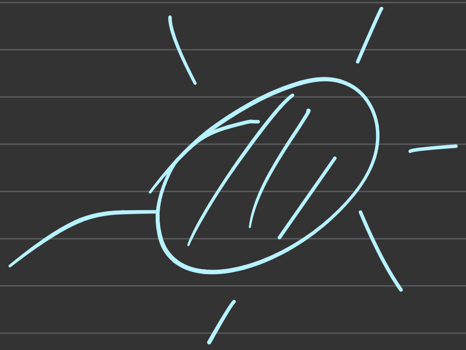
Ex

$$\frac{1}{2}(\partial\phi)^2 + \frac{\lambda_6}{6!}\phi^6 \xrightarrow{\phi \to \frac{1}{\lambda^{1/4}}\phi} \frac{1}{\sqrt{\lambda_6}}\left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{6!}\phi^6\right)$$

$$\sim \lambda_6^{\frac{I-V}{2}} = \lambda_6^{\frac{L-1}{2}} = \frac{1}{\sqrt{\lambda_6}}\lambda_6^{L/2}$$

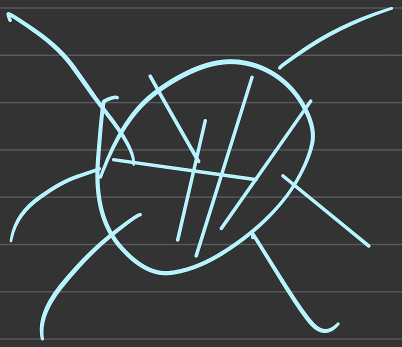
$$\lambda_6 = g_6^4$$

Ex $\frac{1}{2}(\partial\phi)^2 + \frac{\lambda_5}{5!}\phi^5$ $\lambda_5 \sim g_5^3$



$$\simeq (g_5^2)^{L-1} = \lambda_5^{\frac{2}{3}(L-1)}$$

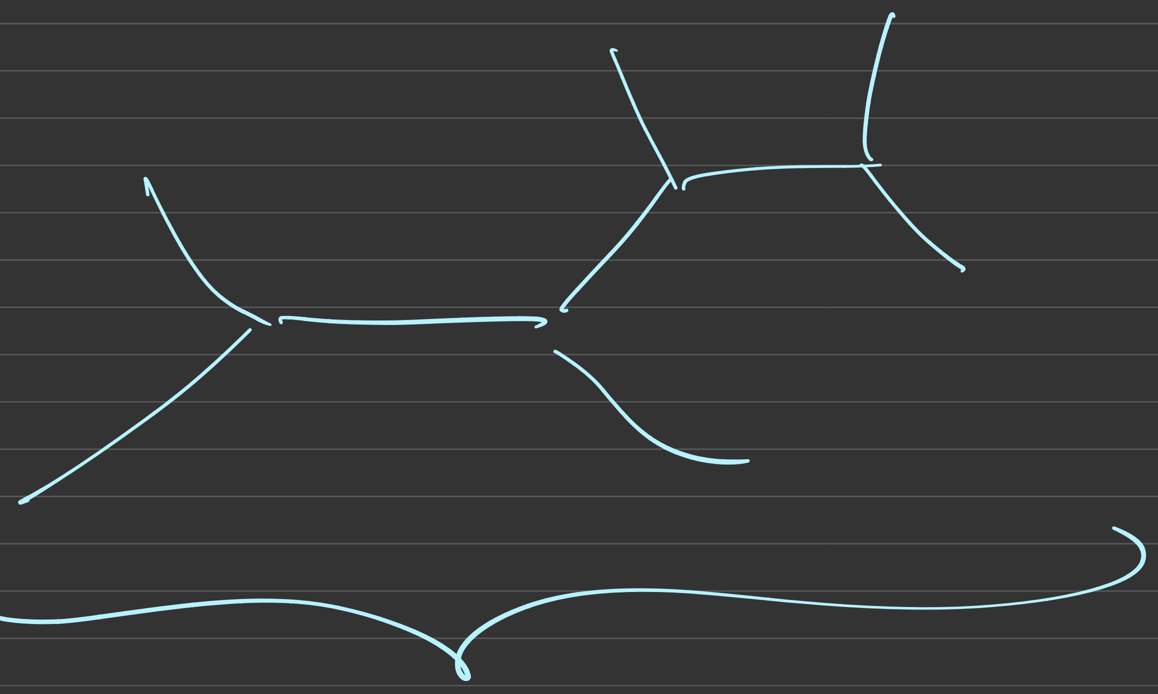
Ex $\frac{1}{2}(\partial\phi)^2 + \frac{\lambda_n}{n!}\phi^n$ $\underline{\lambda_n \sim g_n^{n-2}}$



$$\sim (g_n^2)^{L-1} = (\lambda_n)^{\frac{2(L-1)}{n-2}}$$

$$g_n^{n-2} \quad \Phi_n$$

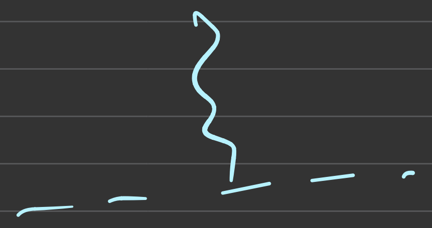
Ex $g\phi^3$

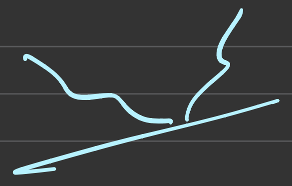


n-legs

$\sim g^{n-2}$

Ex scalar edynamics



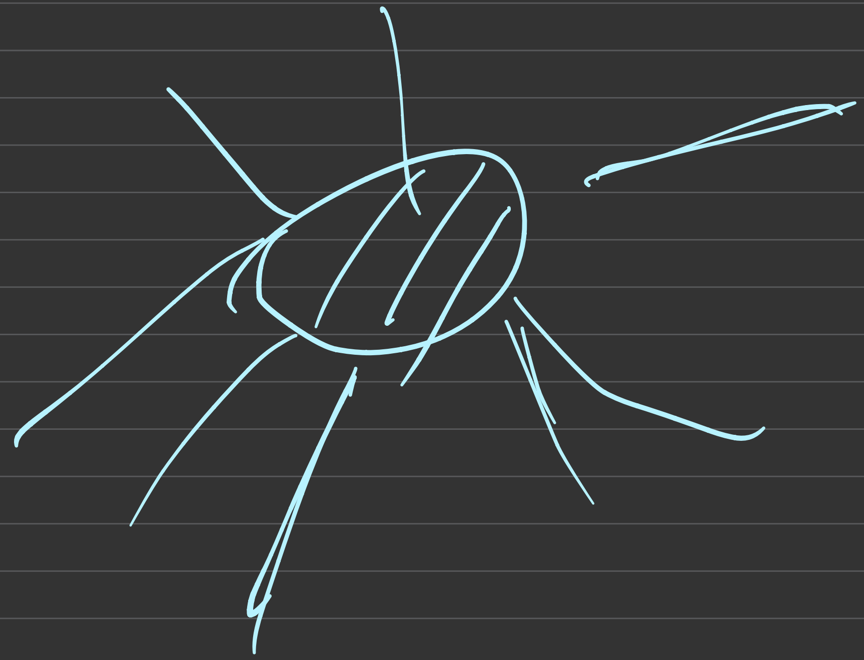


$$\left.\begin{array}{l} e \\ \\ e^2 \end{array}\right\} e^{n-2}$$

- Ex in $g\phi^3$ $\qquad \mathcal{M} = \Pi\left(\frac{p^2-m^2}{\sqrt{Z}}\right) G$

$$\mathcal{M}_{E\text{-legs}} \sim \frac{1}{g^E} \cdot g^{2E} \, (g^2)^{I-V} \sim g^{E+2L-2}$$

$$= \left(g^{E-2}\right) g^{2L}$$



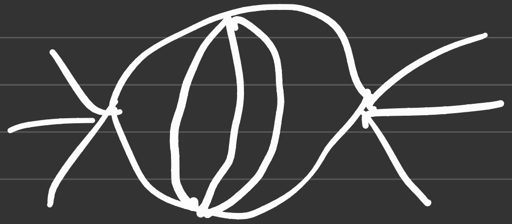
$$\langle \phi \phi \rangle \sim \frac{g^2}{p^2} \xrightarrow{\text{1st}} g \sim \sqrt{Z}$$

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \lambda_5 \phi^5$$

$\mathcal{M}_{3\to3} =$

1-loop

$+$



4-loops

△ The Renormalization Group

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{CT}$$

Ex ϕ^4

$$\mathcal{L}_R = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

$$\mathcal{L}_{CT} = \frac{\delta Z}{2}(\partial\phi)^2 + \frac{1}{2}\overbrace{\left[(m^2+\delta m^2)(1+\delta Z) - m^2\right]}^{\Delta m^2}\phi^2 + \frac{\overbrace{(\lambda+\delta\lambda)(1+\delta Z)^2 - \lambda}^{\Delta\lambda}}{4!}\cdot\phi^4$$

$$\Rightarrow \quad \mathcal{L}_R + \mathcal{L}_{CT} = \frac{Z}{2}(\partial\phi)^2 + \frac{Z}{2}m_0^2\phi^2 + \frac{Z^2}{4!}\lambda_0\phi^4$$

$$\equiv \frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{1}{4!}\lambda_0\phi_0^4$$

$$Z \equiv 1 + \delta Z \qquad m_0^2 \equiv m^2 + \delta m^2 \qquad \lambda_0 \equiv \lambda + \delta\lambda$$

$$\phi_0 \equiv \sqrt{Z}\,\phi$$

Dim Reg

$$\lambda_0 = \lambda \mu^{\varepsilon}\left[1 + \frac{a_1(\lambda)}{\varepsilon} + \frac{a_2(\lambda)}{\varepsilon^2} + \cdots\right]$$

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$$\lambda_0 = \lambda\left[1 + \tilde{a}_1(\lambda)\ln\frac{\Lambda}{\mu} + \tilde{a}_2(\lambda)\ln^2\frac{\Lambda}{\mu} + \cdots\right]$$

$$a_\ell \sim \ell\text{-loop} + (\ell+1)\text{-loop} + \cdots$$

$$= a_{\ell,\ell}\,\lambda^{\ell} + a_{\ell,\ell+1}\,\lambda^{\ell+1} + \cdots$$

- Similarly $$m_0^2 = m^2 \left[1 + \frac{b_1(\lambda)}{\varepsilon} + \frac{b_2(\lambda)}{\varepsilon^2} + \dots \right]$$

$$Z = \left[1 + \frac{c_1(\lambda)}{\varepsilon} + \frac{c_2(\lambda)}{\varepsilon^2} + \dots \right]$$

with $$b_\ell \sim \lambda^\ell + \dots$$

$$c_\ell \sim \lambda^\ell + \dots$$

- Basic structure fixed by dimensional analysis

Ex

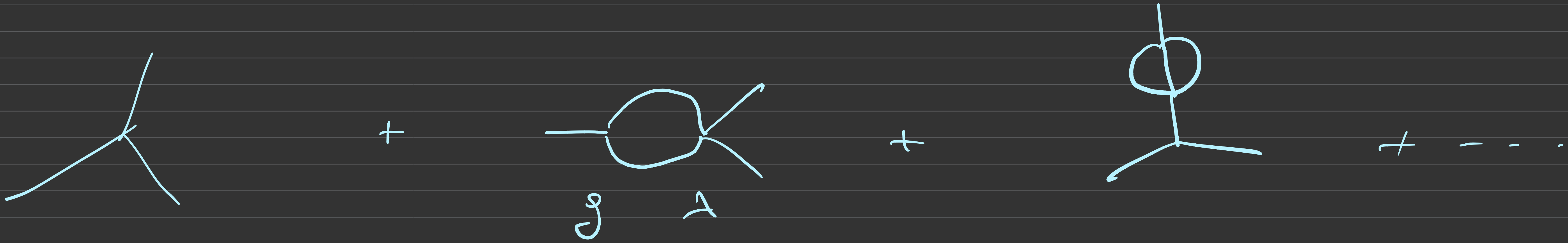
$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \eta\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{3!}g\phi^3 + \frac{1}{4!}\lambda\phi^4$$

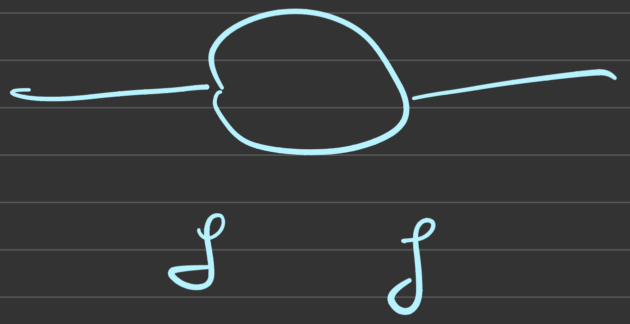
$$g_0 = g\mu^{2-D/2}\left[1 + \frac{d_1(\lambda)}{\varepsilon} + \frac{d_2(\lambda)}{\varepsilon^2} + \cdots\right]$$

$$m_0^2 = m^2\left[1 + \frac{b_1}{\varepsilon} + \cdots\right] + g^2\left[\frac{h_1(\lambda)}{\varepsilon} + \frac{h_2(\lambda)}{\varepsilon^2} + \cdots\right]$$

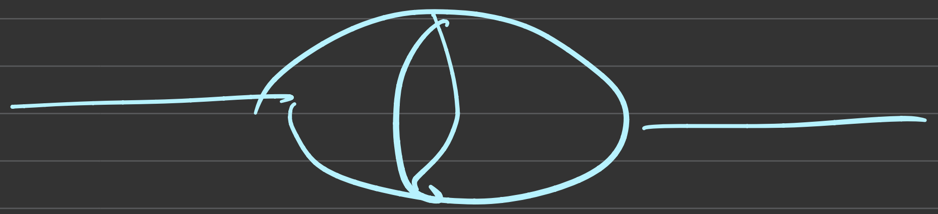
$$\eta_0 = \eta\left[1 + \frac{e_1(\lambda)}{\varepsilon} + \cdots\right] + g m^2\left[\frac{f_1(\lambda)}{\varepsilon} + \cdots\right] + g^3\left[\frac{m_1(\lambda)}{\varepsilon} + \cdots\right]$$

$$g_0 = g\,\mu^{2-D/2}\left[1 + \frac{d_1(\lambda)}{\varepsilon} + \frac{d_2(\lambda)}{\varepsilon^2} + \dots\right]$$

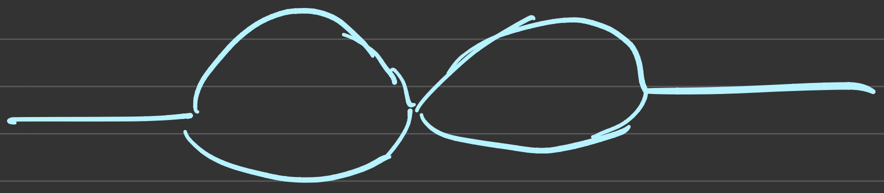




$\sim g^2 \sim$



$g^2\lambda^2$



$g^2\lambda$

- Amplitudes & Correlators $\equiv \mathcal{O}$

$$\mathcal{O} = \lambda^{q}\left[\mathcal{O}_0 + \mathcal{O}_1(\lambda)\ln\frac{E}{\mu} + \mathcal{O}_2(\lambda)\ln^2\frac{E}{\mu} + \dots \right]$$

$$\mathcal{O}_\ell = a_{\ell,\ell}\,\lambda^{\ell} + O(\lambda^{\ell+1})$$

Ex

$$\mathcal{M}(2\to 2) = -\lambda\left[1 + \frac{\lambda}{32\pi^2}\left(3\gamma - 6 + \ln\frac{stu}{(4\pi\mu^2)^3} - i\pi\right) + \dots\right]$$

$$\sim -\lambda - \frac{6\lambda^2}{32\pi^2}\left(c + \ln\frac{E}{\mu}\right) + O(\lambda^3)$$

A) $\mathcal{M} \equiv \mathcal{M}(\lambda, \mu) \quad = -\hat{\lambda} - \frac{6\hat{\lambda}^2}{32\pi^2}\ln E + O(\hat{\lambda}^3)$

$$\uparrow$$

$$\boxed{\cdot\ \hat{\lambda} \equiv \lambda - \frac{6\lambda^2}{32\pi^2}\ln\mu}$$

$$(\lambda', \mu') \sim (\lambda, \mu) \qquad \text{if}$$

$$\lambda' - \frac{6\lambda'^2}{32\pi^2}\ln\mu' = \lambda - \frac{6\lambda^2}{32\pi^2}\ln\mu$$

$$\Rightarrow \quad \lambda' = \lambda - \frac{3\lambda^2}{16\pi^2}\ln\frac{\mu}{\mu'} + O(\lambda^3)$$

B) $\frac{\lambda}{32\pi^2} \ll 1$ not enough to ensure perturbativity if $\ln E/\mu$ sufficiently large

- $\frac{\lambda}{32\pi^2} \ln \frac{E}{\mu} \gtrsim 1 \Longrightarrow$ pert. theory breaks down

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{CT}$$

Renormalization map

- Can this be made more precise ?
- Can we still compute when $\frac{\lambda}{16\pi^2} \ln \frac{E}{\mu} \gtrsim O(1)$?

Basic Idea

$$m^2, \lambda, \mu \longrightarrow \mathcal{L}_0 = \frac{Z}{2}(\partial\phi)^2 + \frac{m_0^2}{2} Z \phi^2 + \frac{\lambda_0}{4!} Z^2 \phi^4$$

$$m'^2, \lambda', \mu' \longrightarrow \mathcal{L}_0' = \frac{Z'}{2}(\partial\phi')^2 + \frac{m_0'^2}{2} Z' \phi'^2 + \frac{\lambda_0'}{4!} Z'^2 \phi'^4$$

Require $\left.\begin{aligned} \lambda_0 &= \lambda_0' \\ m_0 &= m_0' \end{aligned}\right\} \Rightarrow \sqrt{Z}\,\phi = \phi_0$ and $\sqrt{Z'}\,\phi' = \phi_0'$

identical correlators and can be identified

$$\Rightarrow \mathcal{L}_0 = \mathcal{L}_0'$$

$$\begin{tikzcd}
\Rightarrow \quad \mathcal{L} + \mathcal{L}_{CT} \arrow[d] & = & \mathcal{L}' + \mathcal{L}'_{CT} \arrow[d] \equiv \mathcal{L}_0 \\
\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 & & \frac{1}{2}(\partial\phi')^2 + \frac{m'^2}{2}\phi'^2 + \frac{\lambda'}{4!}\phi'^4
\end{tikzcd}$$

⟹ the two construction simply correspond to different splittings of the same <u>bare</u> Lagrangian $\mathcal{L}_0$

- $\forall\ \mu \in \mathbb{R}^+$ I can perform such equivalent construction

$$\lambda\ ,\ m\ ,\ \phi \longrightarrow \lambda(\mu)\ ,\ m(\mu)\ ,\ \phi(\mu)$$

- μ-choice optimal to study physics at $E \sim \mu$ where $\log \frac{E}{\mu} \lesssim O(1)$

- Interpretation

$$\mathcal{L}_0 = \mathcal{L}_R(\mu) + \underset{\substack{\Downarrow \\ \text{compensates quantum fluctuations at } k \gtrsim \mu}}{\mathcal{L}_{CT}(\mu)}$$

△ $$\mathcal{L}_0 = \frac{1}{2}(\partial\phi_0)^2 - \frac{m_0^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$$

- μ-dependence of $\lambda(\mu)$, $m^2(\mu)$, $\phi(\mu)$ implicitly determined by

$$\mu\frac{d}{d\mu}\lambda_0 = 0 \qquad \mu\frac{d}{d\mu}m_0^2 = 0 \qquad \mu\frac{d}{d\mu}\phi_0 = 0$$

$$\Downarrow$$

Renormalization Group Equations

- $\mu \frac{d}{d\mu}\lambda_0 = \mu\frac{d}{d\mu}\left[\lambda\mu^{\varepsilon}\left(1+\frac{a_1}{\varepsilon}+\frac{a_2}{\varepsilon^2}+\cdots\right)\right]$

$$= \left(\overset{\boxed{\mu\frac{d}{d\mu}\lambda}\,\Downarrow}{\hat\beta(\lambda)}+\varepsilon\lambda\right)\left(1+\sum_n\frac{a_n}{\varepsilon^n}\right)+\lambda\hat\beta\left(\sum_n\frac{a_n'}{\varepsilon^n}\right) = 0$$

$$\Rightarrow \quad \hat\beta = -\frac{\varepsilon\lambda+\sum_n\frac{\lambda a_n}{\varepsilon^{n-1}}}{1+\sum_m\frac{(1+\lambda\partial_\lambda)a_m}{\varepsilon^m}}$$

$$= -\left\{\varepsilon\lambda-\lambda^2 a_1' + \frac{\lambda^2}{\varepsilon}\left[-a_2'+a_1'\left(a_1+\lambda a_1'\right)\right]+\cdots\right\}$$

⚠ $\mu \to \mu'$, $\lambda(\mu) \to \lambda(\mu')$, ... $\equiv$ invariance of UV finite observables

$$\Downarrow$$

$$\hat{\beta}(\lambda) \equiv \text{finite}$$

$\Rightarrow$ all $\frac{1}{\varepsilon}$, $\frac{1}{\varepsilon^2}$, ... terms in $\hat{\beta}$ should vanish !!

Ex

$$-a_2' + a_1'(a_1 + \lambda a_1') = 0$$

$a_1 \longrightarrow a_2$ fixed by diff. eq.

$$Q_1 = c_1\lambda + c_2\lambda^2 + \cdots$$

$$Q_1' = c_1 + 2c_2\lambda + \cdots$$

$$Q_2' = (c_1 + 2c_2\lambda)(c_1\lambda + c_2\lambda^2 + c_1\lambda + 2c_2\lambda^2)$$

$$= c_1^2\lambda + O(\lambda^2)$$

$\longrightarrow$ Known

$$\boxed{Q_2 = \frac{1}{2}c_1^2\lambda^2 + O(\lambda^3)}$$

$$\mu \frac{d}{d\mu}\left[\underbrace{\lambda \mu^{\varepsilon}\left(1+\frac{a_1}{\varepsilon}+\frac{a_2}{\varepsilon^2}+\dots\right)}\right]$$

$$\lambda\mu^{\varepsilon}\left(1+\frac{c_1\lambda+\dots}{\varepsilon}+\frac{\frac{1}{2}c_1\lambda^2+\dots}{\varepsilon^2}\dots\right)$$

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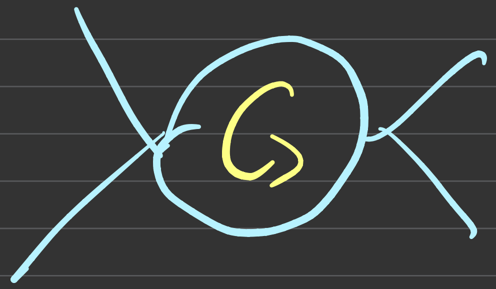
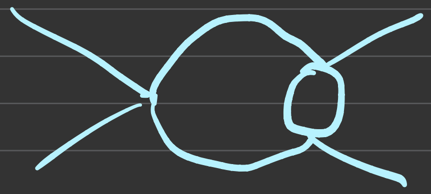


$\log\Lambda$

P.V. $\frac{1}{\varepsilon}\longrightarrow \ln\Lambda$



$$\Longrightarrow \quad \hat{\beta}(\lambda) = -\varepsilon\lambda + \lambda^2 a_1'$$

$$\equiv -\varepsilon\lambda + \beta(\lambda)$$

- For renormalized quantities $\varepsilon \to 0$ smooth

$$\Longrightarrow \quad \boxed{\mu \frac{d}{d\mu} \lambda(\mu) \overset{d=4}{=} \lambda^2 a_1'(\lambda) \equiv \beta(\lambda)}$$

- Similar story for $m^2(\mu)$:

$$m_0^2 = m^2\left[1+\frac{b_1(\lambda)}{\varepsilon}+\frac{b_2(\lambda)}{\varepsilon^2}+\cdots\right]$$

$$0=\mu\frac{d}{d\mu}m_0^2 = \mu\frac{d}{d\mu}m^2\left[1+\sum_n\frac{b_n}{\varepsilon^n}\right]+m^2\left[\sum_n\frac{b_n'}{\varepsilon^n}\right]\left(-\varepsilon\lambda+\lambda^2 a_1'\right)$$

$$\Rightarrow\quad \mu\frac{d}{d\mu}\ln m^2(\mu) = -\frac{\sum_n\frac{b_n'}{\varepsilon^n}\left[-\varepsilon\lambda+\lambda^2 a_1'\right]}{\left[1+\sum_n\frac{b_n}{\varepsilon^n}\right]}$$

$$\equiv -\gamma_m(\lambda)$$

- $$\phi_0 = \sqrt{Z}\,\phi(\mu)$$

$$Z = \left[1 + \frac{c_1(\lambda)}{\varepsilon} + \frac{c_2(\lambda)}{\varepsilon^2} + \dots \right]$$

$$0 = \mu \frac{d}{d\mu} \phi_0 \Rightarrow \mu \frac{d}{d\mu} \phi = -\gamma(\lambda)\,\phi$$

$$\gamma(\lambda) = \frac{1}{2} \mu \frac{d}{d\mu} \ln Z = \frac{1}{2} \left(\frac{\partial}{\partial \lambda} \ln Z \right) \left(-\varepsilon\lambda + \lambda a_1' \right)$$

Formal solutions of RG eqs.

- $$\frac{d\lambda}{d\ln\mu} = \beta(\lambda) \Longrightarrow \ln\frac{\mu'}{\mu} = \int_{\lambda(\mu)}^{\lambda(\mu')} \frac{d\lambda}{\beta(\lambda)}$$

$$d\ln\mu = \frac{d\lambda}{\beta(\lambda)}$$

- $$\frac{d\ln m^2}{d\ln\mu} = -\gamma_m(\lambda) \Longrightarrow \ln\frac{m^2(\mu')}{m^2(\mu)} = -\int_{\ln\mu}^{\ln\mu'} \gamma_m(\lambda(\tilde{\mu}))\, d\ln\tilde{\mu}$$

$$= -\int_{\lambda(\mu)}^{\lambda(\mu')} \frac{\gamma_m(\lambda)}{\beta(\lambda)}\, d\lambda$$

- $\frac{d}{d\ln\mu}\phi(\mu) = -\gamma(\mu)\,\phi(\mu)$

$$\Longrightarrow \quad Z(\mu',\mu) \equiv e^{-\int_{\lambda(\mu)}^{\lambda(\mu')}\frac{\gamma(\lambda)}{\beta(\lambda)}d\lambda}$$

$$\begin{cases} \frac{d}{d\ln\mu'} Z(\mu',\mu) = -\gamma(\lambda(\mu'))\,Z(\mu',\mu) \\ Z(\mu,\mu) = 1 \end{cases}$$

$$\Longrightarrow \quad \phi(\mu') = Z(\mu',\mu)\,\phi(\mu) \Rightarrow \frac{d\phi(\mu')}{d\ln\mu'} = -\gamma(\mu')\,\phi(\mu')$$

▲ Examples

- $\lambda\phi^4$

$$\cdot\ \lambda_0 = \lambda\mu^{\varepsilon}\left(1 + \frac{a_1}{\varepsilon} + \dots\right)$$

$$\cdot\ a_1 = \frac{3\lambda}{16\pi^2}\lambda + O(\lambda^2)$$

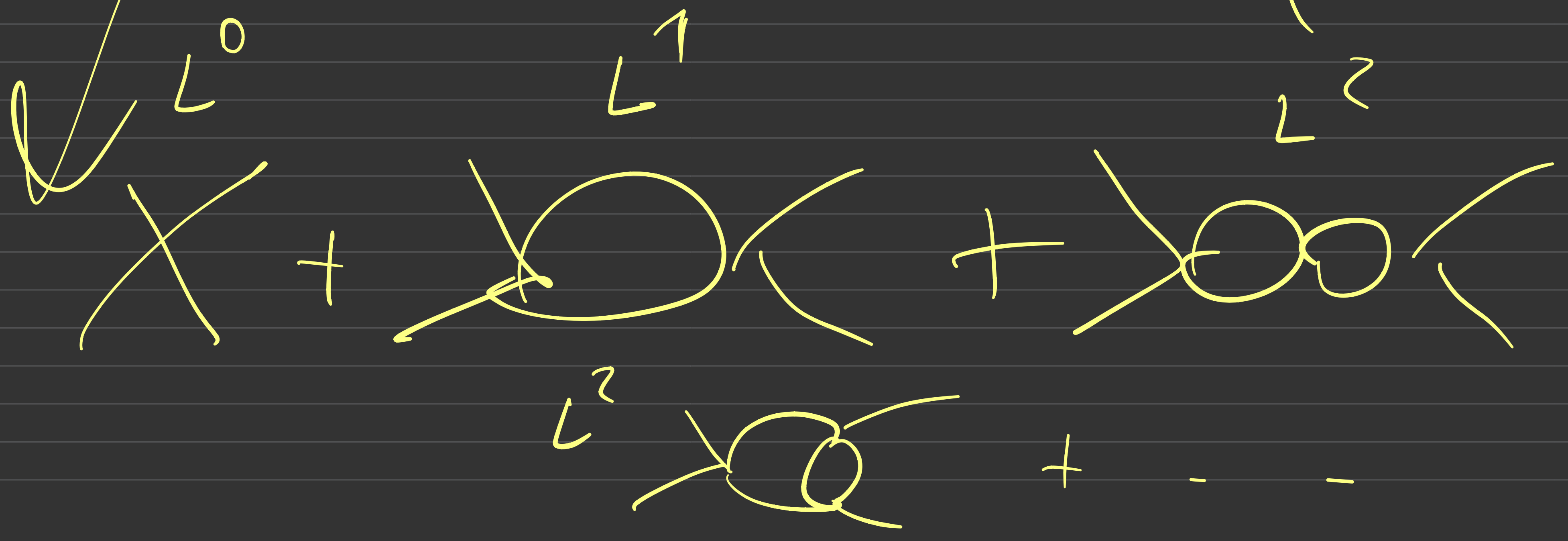
$$\cdot\ \beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3) > 0$$

$$\mu\frac{d}{d\mu}\lambda(\mu) = \frac{3\lambda^2(\mu)}{16\pi^2} \Longrightarrow \boxed{\lambda(\mu') = \frac{\lambda(\mu)}{1 - \frac{3}{16\pi^2}\lambda(\mu)\ln\frac{\mu'}{\mu}}}$$

- $\sigma(E) \approx \frac{1}{E^2} \cdot \lambda(E)^2 \left(1 + c\,\lambda(E) + \dots\right)$

$$\approx \frac{1}{E^2}\,\lambda(E)^2 \approx \frac{1}{E^2}\left[\frac{\lambda(\mu)}{1 - \frac{3\lambda(\mu)}{16\pi^2}\ln E/\mu}\right]^2$$

$$\approx \frac{1}{E^2}\,\lambda(\mu)^2 \left(\sum \left(\frac{3\lambda}{16\pi^2}\right)^n L^n\right)$$



$$L \equiv \ln\frac{E}{\mu}$$

⚠ $$\frac{d\lambda}{d\ln\mu} = c_1\lambda^2 + c_2\lambda^3 + \circ\circ\circ + c_e\lambda^{e+1} + \circ\circ\circ$$

$$\frac{d\ln\lambda}{d\ln\mu} = \lambda\left(c_1 + c_2\lambda + \circ\circ\circ + c_e\lambda^{e-1} + \cdots\right)$$

$$\begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
(\lambda L)^n & \lambda(\lambda L)^n & \lambda^{e-1}(\lambda L)^n \\
\lambda^n L^n & \lambda^{n+1} L^n & \lambda^{e+n-1} L^n
\end{array}$$

△ RG asymptotics of $\lambda\phi^4$ in 4D

$$\lambda(\mu') = \frac{\lambda(\mu)}{1 - \frac{3}{16\pi^2}\lambda(\mu)\ln\frac{\mu'}{\mu}}$$

- IR $\quad \ln \mu'/\mu \longrightarrow -\infty \implies \lambda(\mu') \simeq \dfrac{16\pi^2}{3\ln\frac{\mu}{\mu'}} \longrightarrow 0$

- UV $\quad \dfrac{3\lambda(\mu)}{16\pi^2}\ln\dfrac{\Lambda_*}{\mu} = 1 \implies \lambda(\Lambda_*) = \infty \quad$ Landau Pole

$$\ln\frac{\Lambda_*}{\mu} = \frac{16\pi^2}{3\lambda(\mu)} \longrightarrow \Lambda_* = \mu\, e^{\frac{16\pi^2}{3\lambda(\mu)}}$$

⚠ Landau Pole & bare couplings

$$\ln\frac{\Lambda}{\mu} \nearrow \frac{1}{\varepsilon} \text{ in DR}$$

- Pauli-Villars : $\lambda_0 = \lambda(\mu)\, P\left(\lambda(\mu), \ln\frac{\Lambda}{\mu}\right)$

$$\Downarrow$$

$$\sum_{n=0}^{\infty} p_n(\lambda)\left(\ln\frac{\Lambda}{\mu}\right)^n$$

$p_0 = 1$

$p_n = c\lambda^n + \dots$

- $\Lambda \simeq \mu$: $\lambda_0 = \lambda(\Lambda)\left[1 + a_1\lambda(\Lambda) + a_2\lambda^2(\Lambda) + \dots\right]$

hp weak coupling

$$\lambda_0 \sim \lambda(\Lambda) \simeq \frac{\lambda(\mu)}{1 - \frac{3}{16\pi^2}\lambda(\mu)\ln\frac{\Lambda}{\mu}}$$

$$\swarrow$$

$$\lambda_0 = \lambda(\mu) + \frac{3}{16\pi^2}\lambda(\mu)\ln\frac{\Lambda}{\mu} + \left(\frac{3\lambda(\mu)}{16\pi^2}\ln \Lambda/\mu\right)^2 + \ldots$$

we must work under assumption that $\frac{3\lambda(\mu)}{16\pi^2}\ln \Lambda/\mu \ll 1$

$$\Lambda \ll \Lambda_* \equiv \text{Landau pole scale}$$

Asymptotics of $g\phi^3$ in 6D

$$g_0 = g\,\mu^{\varepsilon/2}\left(1 - \frac{g^2}{64\pi^3}\,\frac{1}{\varepsilon} + \dots\right)$$

$$\mu\frac{d}{d\mu}g_0 = 0 \quad \Longrightarrow \quad \hat{\beta}(g) = -\frac{\varepsilon}{2}\,g - \frac{3g^3}{128\,\pi^3}$$

- 6D $$\frac{dg}{d\ln\mu} = -\frac{3g^3}{128\pi^3} \quad \Longrightarrow \quad \frac{dg^2}{d\ln\mu} = -\frac{3g^4}{64\pi^3}$$

$$g^2(\mu') = \frac{g^2(\mu)}{1+\frac{3}{64\pi^3}\,g^2(\mu)\,\ln\frac{\mu'}{\mu}}$$

- UV : $\ln\frac{\mu'}{\mu} \to \infty \implies g^2(\mu') \sim \frac{64\pi^3}{3\ln\frac{\mu'}{\mu}} \to 0$

Asymptotic Freedom

- IR $g(\mu_*) = \infty$ for $\frac{3}{64\pi^3}\, g^2(\mu) \ln\frac{\mu_*}{\mu} = -1$

$$\boxed{\mu_* = \mu\, e^{-\frac{64\pi^3}{3g^2(\mu)}}}$$

dimensional transmutation

▲ Bare coupling (P. V.)

$$g_0^2 \simeq \frac{g^2(\mu)}{1+\frac{3}{64\pi^3}\, g^2(\mu)\ln\frac{\Lambda}{\mu}}$$

- $\left(g^2 \ln\frac{\Lambda}{\mu}\right)^n$ series of counter-terms makes sense also when $g^2 \ln\frac{\Lambda}{\mu} \to \infty$

$$0 = \left[\sum_i \gamma_i + \mu \frac{d}{d\mu} \right] G$$

- $G = G(p_1, \ldots p_n, \lambda_a(\mu), \mu)$

$\lambda_a \equiv$ all parameters

$\equiv$ all couplings & all masses

$$\boxed{\left(\sum_i \gamma_i + \sum_a \beta_a \frac{\partial}{\partial \lambda_a} + \mu \frac{\partial}{\partial \mu} \right) G = 0}$$