- Loop counting parameters

· It obvious that, for any given observable, its expansion in a power series in the coupling corresponds to a diagrammatic loop expansion. It is worth making this more precise. A fex example will suffice

Ex 1

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{g}{3!}\phi^3$$

$$\phi \to \phi/g$$

$$\mathcal{L} \to \frac{1}{g^2}\left(\frac{(\partial\phi)^2}{2} + \frac{\phi^3}{3!}\right)$$

now $G(p) \propto g^2 \longrightarrow \# = I$

$\text{Vertex} \propto \frac{1}{g^2} \longrightarrow \# = V$

a graph G corresponds to $(g^2)^{I-V}$

$L = I - V + 1 \implies (g^2)^{L-1}$

$L = 0 \longrightarrow \frac{1}{g^2}$ tree

$L = 1 \longrightarrow g^0$ 1-loop

etc

Ex. 2 ϕ^4 $(\partial\phi)^2 + \lambda_4 \phi^4$

$\phi \to \frac{1}{\sqrt{\lambda}} \phi$

$\mathcal{L} \to \frac{1}{\lambda_4} \left((\partial\phi)^2 + \phi^4 \right)$

λ_4 is the loop counting : $\lambda_4 \sim g^2$

Ex 3 ϕ^6 $(\partial\phi)^2 + \lambda_6 \phi^6$

$\phi \to \frac{1}{(\lambda_6)^{1/4}} \phi$

$\mathcal{L} \to \frac{1}{\sqrt{\lambda_6}} \left((\partial\phi)^2 + \phi^6 \right)$

loop counting $\longrightarrow$ $(\lambda_6)^{1/2}$

· given only λ_6 appears in the pert. expansion the graph series advances in two loop steps.

Comparing to ϕ^3 $\qquad \lambda_6 \sim g^4$

In general $\phi^n \longrightarrow g^{n-2}\phi^n$

- Notice all this makes sense in ϕ^3 $\quad \mathcal{M}_{n\text{-legs}} \sim \frac{1}{g^E}\, g^{2E}\, g^{2I-2V}$

$$\sqrt{E}$$

$$= g^{E+2L-2} = \left(g^{E-2}\right) g^{2L}$$

$L=0$

$$g^{E-2}$$

▲ The Renormalization Group

The renormalization procedure leads order by order in perturbation theory to construct a Lagrangian yelding finite n-point correlators. $\mathcal{L}$ takes the form

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{CT}$$

Ex in ϕ^4 $\quad \mathcal{L}_R = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$

$$\mathcal{L}_{CT} = \frac{\delta Z}{2}(\partial\phi)^2 + \left[(m^2+\delta m^2)(1+\delta Z) - m^2\right]\frac{\phi^2}{2}$$

$$\frac{(\lambda+\delta\lambda)(1+\delta Z)^2 - \lambda}{4!}\phi^4$$

$$\mathcal{L} = \frac{Z}{2}(\partial\phi)^2 + \frac{Z}{2}m_0^2\phi^2 + \frac{Z^2\lambda_0}{4!}\phi^4$$

$$\equiv \frac{1}{2}(\partial\phi_0)^2 + \frac{m_0^2}{2}\phi_0^2 + \frac{\lambda_0\phi_0^4}{4!}$$

$$M_0^2 \equiv m^2 + \delta m^2 \qquad \lambda_0 \equiv \lambda + \delta\lambda \qquad \phi_0 \equiv \sqrt{Z}\,\phi$$

bare parameters $\qquad$ bare field

- Notice the <u>complementarity</u>:

$\mathcal{L}$ is <u>physical</u> but <u>singular</u>, $\mathcal{L}_R$ is <u>finite</u> but <u>unphysical</u> as the split $\mathcal{L}_R + \mathcal{L}_{CT}$ clearly possesses a degree of arbitrariness.

In mass independent schemes the bare quantities can be written as power series in the couplings, in accordance to Weinberg's theorem

Ex λ $\qquad$ Dim Reg $\quad \lambda_0 = \lambda\mu^{\varepsilon}\left[1 + \frac{a_1(\lambda)}{\varepsilon} + \frac{a_2(\lambda)}{\varepsilon^2} + \dots\right]$

P.V. $\quad \lambda_0 = \lambda\left[1 + \tilde{a}_1(\lambda)\ln\frac{\Lambda}{\mu} + \tilde{a}_2(\lambda)\ln^2\frac{\Lambda}{\mu} + \dots\right]$

with $a_e(\lambda)$ starting at e-loops:

$$a_e(\lambda) = a_{e,e}\,\lambda^{e} + a_{e,e+1}\,\lambda^{e+1} + \dots$$

- in $g\phi^3$ they same will hold with g^2 playing the rôle of loop counting

- In mass indep schemes

$$m_0^2 = m^2\left[1 + \frac{b_1(\lambda)}{\varepsilon} + \frac{b_2(\lambda)}{\varepsilon^2} + \dots\right]$$

$$Z_1 = \left[1 + \frac{c_1(\lambda)}{\varepsilon} + \frac{c_2(\lambda)}{\varepsilon^2} + \dots\right]$$

with same property $b_e \sim \lambda^e + \dots$

$c_e \sim \lambda^e + \dots$

- the point is that from the point of view of UV divergences $m^2\phi^2$ can be treated like any other inte raction. then dimensional analysis fixes the form of m_0^2/m^2.

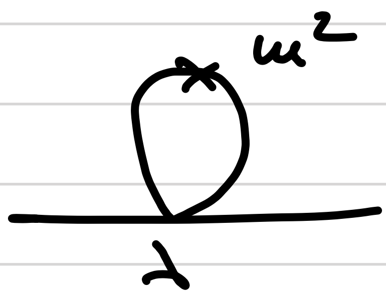
We could even consider adding $g\,\phi^3$

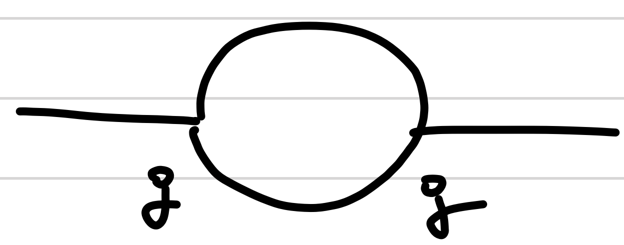
$[g] = 1$ in 4D, in which case

$$g_0 = g\mu^{2-D/2}\left[1 + \frac{G_1(\lambda)}{\varepsilon} + \frac{G_2(\lambda)}{\varepsilon^2} + \dots\right]$$

$$-3\left(\frac{D}{2}-1\right) + D = 3 - D/2 = 1 + 2 - D/2$$

$$m_0^2 = m^2\left[1 + \frac{b_1}{\varepsilon} + \dots\right] + g^2\left[\frac{d_1(\lambda)}{\varepsilon} + \frac{d_2(\lambda)}{\varepsilon^2} \dots\right]$$





• Any physical amplitude will similarly result written as a power series in λ at $\ln\mu$ ($\ln E/\mu$ with E any combination of kinematic invariants):

$$\mathcal{O} = \lambda^q \left[\mathcal{O}_0 + \mathcal{O}_1(\lambda) \ln\frac{E}{\mu} + \mathcal{O}_2(\lambda) \ln^2\frac{E}{\mu} + \dots \right]$$

$$\mathcal{O}_\ell = a_\ell \lambda^\ell + O(\lambda^{\ell+1})$$

For instance for $\mathcal{M}(2\to 2)$ we found at 1-loop ($m=0$)

$$\mathcal{M} = -\lambda \left[1 + \frac{\lambda}{32\pi^2} \left(3\gamma - 6 + \ln\frac{stu}{(4\pi\mu^2)^3} - i\pi \right) \dots \right]$$

$$\sim -\lambda - \frac{\lambda^2}{32\pi^2} \ln\frac{stu}{(4\pi\mu^2)^3} + \text{const}$$

neglecting

$$\sim -\lambda - \frac{6\lambda^2}{32\pi^2} \ln\frac{E}{\mu}$$

Comments

A) $\mathcal{M}$ seemingly depends on 2 parameters (λ, μ). However, consistently neglecting $O(\lambda^3)$, only one combination appears:

$$\lambda + \frac{6\lambda^2}{32\pi^2} \ln \mu = \hat{\lambda}$$

$$\Rightarrow \mathcal{M} \sim -\hat{\lambda} - \frac{6\hat{\lambda}^2}{32\pi^2} \ln E$$

$\Rightarrow$ equivalently: starting with $\lambda' \neq \lambda$, $\mu' \neq \mu$ such that however

$$\lambda' + \frac{6\lambda'^2}{32\pi^2} \ln \mu' = \lambda + \frac{6\lambda^2}{32\pi^2} \ln \mu \equiv \hat{\lambda}$$

we obtain the same result

$\Rightarrow$ this suggest (λ, μ) is a redundant pair

B) $\lambda \ll$ does not seem enough to carry on perturbation theory if E/μ is so large that $\frac{3\lambda \ln E/\mu}{16\pi^2} \gtrsim 1$

A et B seem both consistent with the idea that μ is associated to a split $\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{CT}$ which is optimized to compute a certain class of observables (correlators with momenta of order μ).

- Can this picture be made precise (all orders)?
- Can calculability be recovered when $\frac{\lambda}{16\pi^2} \ln E/\mu \sim O(1)$?

Both question have affirmative answers. The required methodology is the Renormalization Group (RG).

· The idea is very simple. Once employed for parameters m^2, λ, μ the renormalization procedure could be carried out for another, different set, m'^2, λ', μ'. That would define a Lagrangian

$$\mathcal{L} = \frac{Z'}{2}(\partial\phi')^2 + \frac{m_0'^2 Z'\phi'^2}{2} + \frac{\lambda_0' Z'^2 \phi'^4}{4!}$$

with the primed parameters Z', m_0', λ_0' having the same functional dependence on μ', λ', m' as $Z, m_0 \lambda_0$ on μ, λ, m.

Now, if we <u>require</u> $\lambda_0 = \lambda_0'$, $m_0 = m_0'$ up to a field rescaling the two constructions define precisely the same QFT.

Moreover, once this request is made $\phi_0 \equiv \sqrt{Z}\phi$ and $\phi_0' \equiv \sqrt{Z'}\phi'$.

will have exactly the same correlators $\Longrightarrow$ they can be viewed as the same field.

These two procedures, must thus simply correspond to two different splits

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{CT} \equiv \mathcal{L}_R' + \mathcal{L}_{CT}'$$

or, in other words, to two different renormalization schemes for the same QFT.

- We can imagine doing so for a continuum of μ values. The three conditions $\lambda_0, m_0, \phi_0 \equiv \text{const}$ as μ varies, thus define 3 μ dependent quantities: $\lambda(\mu)$, $m(\mu)$, $\phi(\mu)$.

Base on what we have already seen

the μ-dependent, or running, quantities define the "optimal" split when considering correlators that involve just one overall energy scale of order μ.

- Based on the above, the running parameters are implicitly defined by

$$\mu \frac{d}{d\mu} \lambda_0 = 0 \qquad \mu \frac{d}{d\mu} m_0^2 = 0 \qquad \mu \frac{d}{d\mu} \phi_0 = 0$$

which we will now study.

- $\mu \frac{d}{d\mu} \lambda = \hat{\beta}(\lambda)$

$$\mu \frac{d}{d\mu}\left[\lambda \mu^{\varepsilon}\left(1 + \frac{a_1}{\varepsilon} + \frac{a_2}{\varepsilon^2} + \dots\right)\right] = 0$$

$$(\hat{\beta} + \varepsilon\lambda)\left(1 + \sum_j \frac{a_j}{\varepsilon^j}\right) + \lambda\hat{\beta}\left(\sum_j \frac{a'_j}{\varepsilon^j}\right) = 0$$

$$\hat{\beta}\left(1+\sum_j \frac{(1+\lambda\partial_\lambda)a_j}{\varepsilon^j}\right) = -\varepsilon\lambda - \sum_j \frac{\lambda a_j}{\varepsilon^{j-1}}$$

$\Downarrow$ formally

$$\hat{\beta} = -\frac{\varepsilon\lambda + \sum_j \frac{\lambda a_j}{\varepsilon^{j-1}}}{1+\sum_j \frac{(1+\lambda\partial_\lambda)a_j}{\varepsilon^j}}$$

$$= -\Big[\varepsilon\lambda - \lambda^2\partial_\lambda a_1 + \frac{1}{\varepsilon}\Big(\cancel{\lambda a_2} - \cancel{\lambda a_2} - \lambda^2 a_2'$$
$$+ \lambda(a_1 + \lambda a_1')^2\Big) + O\Big(\frac{1}{\varepsilon^2}\Big)\Big]$$

$$= -\varepsilon\lambda + \lambda^2 a_1' + \frac{1}{\varepsilon}\Big[\lambda(a_1+\lambda a_1')^2 - \lambda a_2'\Big] + \dots$$

Now any physical observable, like the S-matrix, must be invariant under the change

$\mu \to \mu'$ $\lambda \to \lambda'$ .. etc

$\Rightarrow$ the change of λ needed to compensate the change of μ must be finite

$\Longrightarrow$ $\hat{\beta}$ must be finite.

We immediately conclude that the structure of counterterms must be such that the coefficient of all $1/\varepsilon$ powers must vanish identically.

This seems a truly remarkable result: Not only Weinberg theorem states the existence of $(\log)^\ell$ or $(1/\varepsilon)^\ell$ at ℓ-loops but the self-consistency of the whole procedure implies that $a_2, a_3, a_4 \ldots$ are all determined by a_1: log divs fully determine $(\log)^\ell$ divs $\forall \ell$.

Not fully surprising as $(\log)^\ell$ can be viewed as ℓ-loops within one another $\Rightarrow$ log divs factorize.

in the end we have

$$\hat{\beta}(\lambda) = -\varepsilon\lambda + \lambda^2 a_1'$$
$$\equiv -\varepsilon\lambda + \beta(\lambda)$$

when considering renormalized quantities we can safely take the $\varepsilon \to 0$ limit

$$\Longrightarrow \quad \mu \frac{d}{d\mu}\lambda \overset{d=4}{=\!=} \beta(\lambda) = \lambda^2 \partial_\lambda a_1$$

- We can do precisely the same for m^2: $\quad \mu \frac{d}{d\mu} m^2 = -\gamma_m(\lambda)\, m^2$

<u>Ex</u> find $\gamma_m(\lambda)$ in terms of the b's and a's

- finally $\phi_0 = \sqrt{Z}\,\phi$

$$0 = \mu \frac{d}{d\mu}\phi_0 \Longrightarrow \mu \frac{d}{d\mu}\phi(\mu) = -\frac{1}{2}\left(\mu \frac{d}{d\mu}\ln Z\right)\phi$$

$$\mu \frac{d}{d\mu} \phi(\mu) \equiv -\gamma(\lambda)\phi$$

$$\gamma(\lambda) \equiv \frac{1}{2} \mu \frac{d}{d\mu} \ln Z = \frac{1}{2} \frac{d}{d\lambda} \ln Z \cdot (-\varepsilon\lambda + \hat{\beta})$$

Ex find γ in terms of a's, c's

- Formal solutions

- $$\frac{d\lambda}{d\ln\mu} = \beta(\lambda)$$

$$\Rightarrow \ln \mu'/\mu = \int_{\lambda(\mu)}^{\lambda(\mu')} \frac{d\lambda}{\beta(\lambda)}$$

- $$m^2(\mu') = m^2(\mu)\, e^{-\int_{\ln\mu}^{\ln\mu'} \gamma_m(\lambda(\tilde{\mu}))\, d\ln\tilde{\mu}}$$

$$= m^2(\mu)\, e^{-\int_{\lambda(\mu)}^{\lambda(\mu')} \frac{\gamma_m(\lambda)}{\beta(\lambda)} d\lambda}$$

- $$\zeta(\mu',\mu) = e^{-\int_{\lambda(\mu)}^{\lambda(\mu')} \frac{\gamma(\lambda)}{\beta(\lambda)} d\lambda}$$

$$\begin{cases} \mu' \dfrac{d}{d\mu'} Z(\mu', \mu) = -\gamma(\lambda(\mu')) Z(\mu', \mu) \\ Z(\mu, \mu) = 1 \end{cases}$$

$$\Longrightarrow \quad \phi(\mu') = Z(\mu', \mu)\, \phi(\mu)$$

Comment on nomenclature: the change of μ can intuitively (and also more precisely) be associated to a dilation That is to the abelian group of dilations hence the name Renormalization Group (RG)

• Example of RG evolution

$\lambda\phi^4$ in 4D

from previous results

$$\lambda_0 = \lambda\mu^\varepsilon\left(1 + \frac{a_1(\lambda)}{\varepsilon} + \dots\right)$$

$$a_1 = \frac{3\lambda}{16\pi^2} + O(\lambda^2)$$

$$\Rightarrow \beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3)$$

$\Rightarrow$ as long as λ is weak $\beta > 0$

· Solution at 1-loop

$$\frac{16\pi^2}{3}\frac{d\lambda}{\lambda^2} = d\ln\mu \qquad \frac{16\pi^2}{3}\left(\frac{1}{\lambda(\mu)} - \frac{1}{\lambda(\mu')}\right) = \ln\frac{\mu'}{\mu}$$

$$\frac{1}{\lambda(\mu')} = \frac{1}{\lambda(\mu)} - \frac{3}{16\pi^2}\ln\mu'/\mu$$

$$\lambda(\mu') = \frac{\lambda(\mu)}{1 - \frac{3}{16\pi^2}\lambda(\mu)\ln\mu'/\mu}$$

- the solution resums effects of the type $\lambda \log = \lambda L$
- one can easily check that ℓ-th loop order resum terms $\lambda^{\ell-1}(\lambda L)^n$ $\forall n$

Indeed

$$\frac{d\lambda}{dL} = c_1 \lambda^2 + c_2 \lambda^3 + \dots c_\ell \lambda^{\ell+1}$$

$$= \lambda^2 (\underbrace{c_1}_{\Downarrow} + c_2 \lambda + \dots c_\ell \lambda^{\ell-1} + \dots)$$

$$(\lambda L)^n \qquad \lambda (\lambda L)^n \qquad \lambda^{\ell-1} (\lambda L)^n$$

$$\frac{d(\lambda/\lambda_0)}{\lambda_0 dL} = \frac{\lambda^2}{\lambda_0^2}\left[c_1 + c_2\lambda + c_3\lambda^2 + \dots\right]$$

$\lambda/\lambda_0 = y \qquad \lambda_0 L = x$

$$\frac{dy}{dx} = y^2\left[c_1 + c_2\lambda_0 y + c_3\lambda_0^2 y^2 \dots\right]$$

$$y(x) = F_0(x) + \lambda_0 F_1(x) + \lambda_0^2 F_2(x) + \dots$$

- $$\lambda(\mu') = \frac{\lambda(\mu)}{1 - \frac{3\lambda(\mu)}{16\pi^2}\ln \mu'/\mu}$$

- $$\lim_{\mu' \to \infty} \lambda(\mu') \sim \frac{16\pi^2}{3\ln \mu/\mu'} \longrightarrow 0$$

theory is free in R

- at $\frac{3\lambda}{16\pi^2}\ln\frac{\mu_L}{\mu} = 1 \qquad \lambda(\mu_L) \to \infty$

Landau Pole : perturbation theory

breaks down in the UV

- We constructed the series of counterterms assuming nothing would go wrong for $\lambda L(x) \gtrsim 1$, but this is not true in ϕ^4. For such theory we are force to assume the UV cut-off cannot be sent to ∞ otherwise the c.T. series develops a singularity.

• <u>Landau pole</u> <u>and</u> <u>bare</u> <u>couplings</u>

Consider for definiteness the case of Pauli-Villars in $\lambda\phi^4$.

We have

$$\lambda_0 = \lambda(\mu)\ \underbrace{P(\lambda(\mu), \ln \Lambda/\mu)}_{\text{infinite series}}$$

imagine now taking $\mu \simeq \Lambda$

$$\lambda_0 = \lambda(\Lambda)\left[1 + a_1 \lambda(\Lambda) + a_2 \lambda^2(\Lambda) \dots\right]$$

no-logs

$\Rightarrow \quad \lambda_0 = \lambda(\Lambda)$ up to a change of scheme

$\Rightarrow \lambda(\Lambda) \to \infty$ is problematic for a perturbative definition of the bare Action.

ϕ^3 in 6D

$$g \to \mu^{\varepsilon/2} g\left(1 - \frac{g^2}{64\pi^3}\frac{1}{6-d} + \dots\right)$$

$$\mu\frac{d}{d\mu} = 0 \qquad \varepsilon/2\, g\left(1 - \frac{g^2}{64\pi^3}\frac{1}{\varepsilon} + \dots\right) +$$

$$+ \beta\left(1 - \frac{3g^2}{64\pi^3}\frac{1}{\epsilon}\right) = 0$$

$$\beta = -\frac{\varepsilon}{2} g\left(1 - \frac{g^2}{64\pi^3}\frac{1}{\epsilon} + \frac{3g^2}{64\pi^3}\frac{1}{\varepsilon}\right)$$

$$= -\varepsilon/2\, g - \frac{3g^3}{128\pi^3} + \dots$$

OK

$$\mu\frac{d}{d\mu} g = -\frac{3g^3}{128\pi^3}$$

$$\mu\frac{d}{d\mu} g^2 = -\frac{3g^4}{64\pi^3} < 0 \qquad g^2 = \lambda$$

$$g^2(\mu') = \frac{g^2(\mu)}{1 + \frac{3}{64\pi^3} g^2(\mu) \ln \mu'/\mu}$$

We have here the opposite behaviour

- $\lim_{\mu' \to \infty} g^2(\mu') \sim \frac{64\pi^3}{3 \ln \mu'/\mu} \longrightarrow 0$

- $g^2(\mu') \to \infty$ for

$$\ln \mu^*/\mu = -\frac{64\pi^3}{3 g^2(\mu)}$$

$$\boxed{\mu^* = \mu \, e^{-\frac{64\pi^3}{3 g^2(\mu)}}}$$

Strong coupling and loss of perturbativity occurr at an exponentially small IR scale. But this does not represent any problem in the microscopic definition of the theory.

In fact by the same argument as previously

$$g_0 \sim g(\Lambda) = \frac{g(\mu)}{1 + \frac{3}{64\pi^3} g^2(\mu) \ln \frac{\Lambda}{\mu}}$$

$$\Downarrow$$

the series of counterterms, which we constructed assuming $g^2(\mu) \ln \frac{\Lambda}{\mu} \ll 1$ can be analytically continued at $g^2 \ln \Lambda/\mu \gg 1$ (and positive) without encountering any singularity.

$g_0 \sim$ geometric series in $g^2 \ln \Lambda/\mu$, radius of convergence finite, but singularity not problematic cause it

lies in the infrared ($\lambda \to 0$)
while our worry is $\lambda \to \infty$

- In fact in $g\phi^3$

$$g(\lambda) \sim \frac{64\pi^3}{3 \ln \lambda/\mu^*} \xrightarrow{\lambda \to \infty} 0$$

theory is free in limit $\lambda \to \infty$
(asymptotic freedom)