⚠ Multiparticle states & cluster principle : Reading

- Coleman : 7.4, 14.1, 14.2
- Weinberg : 10.2, 10.3
- LSZ original paper

Multiparticle states & cluster principle

- particle wave packet: $|\varphi\rangle \equiv \int d\Omega_p \, \varphi_\sigma(p) \, |p, \sigma\rangle$

$$\langle 0| \mathcal{O}_A(x) |\varphi\rangle = \int d\Omega_p \, \varphi_\sigma(p) \, e^{-ipx} \, \psi_{A,\sigma}(p) \equiv \hat{\psi}_A(x)$$

- $\varphi_\sigma(p)$ <u>smooth</u> $\longrightarrow$ $\psi_A(x)$ <u>localized</u>

- natural interpretation: $|\varphi\rangle$ =

"1-particle blob"

Notice $\underline{\langle \varphi_1 | \varphi_2 \rangle} = \int d\Omega_1 \, d\Omega_2 \; \varphi_1^*(k_1)\, \varphi_2(k_2) \, \langle k_1 | k_2 \rangle$

$$= \int d\Omega_1 \, d\Omega_2 \; \varphi_1^*(k_1)\, \varphi_2(k_2) \, (2\pi)^3 \, 2k_1^0 \, \delta^3(\vec{k}_1 - \vec{k}_2)$$

$$= \int d\Omega_k \; \varphi_1^*(k)\, \varphi_2(k) \equiv (\varphi_1, \varphi_2)$$

- Conversely: define "smeared" operator $(\Box + M^2)\chi = 0$

$$\mathcal{O}_\chi(t) \equiv \int d^3x \; \mathcal{O}(t,x)\, \chi^*(t,x) \quad \text{with} \quad \chi(t,x) = \int d\Omega_p \, 2E_p \, e^{-ipx} \, \tilde{\chi}(\vec{p})$$

$$E_p = p_0 \equiv \sqrt{\vec{p}^{\,2} + M^2}$$

$$\Rightarrow \langle \chi(t) | k \rangle \equiv \langle 0 | \mathcal{O}_\chi(t) | k \rangle =$$

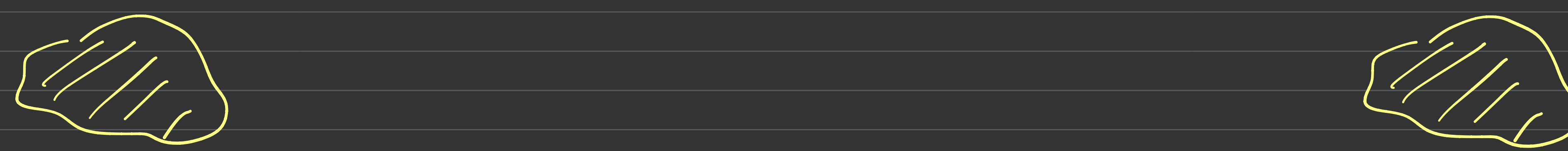
$$\int d^3x \, d\Omega_p \, 2E_p \, \langle 0 | \mathcal{O}(0) | k \rangle \, e^{-ikx} \, e^{ipx} \, \chi(p)^*$$

$$= \langle 0 | \mathcal{O}(0) | k \rangle \, \chi(k)^* \equiv Z \, \chi(k)^*$$

$$\langle \chi | \varphi \rangle = Z \int \chi(k)^* \, \varphi(k) \, d\Omega_k$$

• Cluster Principle

1. $\exists$ states featuring space seperated 1-particle blobs



$\Rightarrow$ given $|p_r, r, \sigma\rangle$

$p_r^2 \equiv M_r^2 \qquad s_r, Q_r, \ldots \qquad \sigma \rightarrow -s_r, \ldots, s_r$

Hilbert space must include "multi particle states"

$$|(p_1, r_1, \sigma_1); (p_2, r_2, \sigma_2); \ldots \ldots (p_n, r_n, \sigma_n)\rangle$$

$$\propto (p_1, r_1, \sigma_1) \otimes (p_2, r_2, \sigma_2) \;----\; \otimes (p_n, r_n, \sigma_n)$$

- multiparticle blobs:

$$|\varphi_1 \cdots \varphi_n\rangle \equiv \int \prod_{i=1}^{r} \left(d\Omega_i \, \varphi^i_{\sigma_i}(p_i) \right) |(p_1, r_1, \sigma_1) \cdots (p_n, r_n, \sigma_n)\rangle$$

2. Matrix elements factorize:

$$\langle 0| \mathcal{O}_{\alpha_1 A_1}(x_1) \cdots \mathcal{O}_{\alpha_n A_n}(x_n) | \varphi_1 \cdots \varphi_n \rangle = \sum_{\pi} \prod_{i=1}^{n} \langle 0 | \mathcal{O}_{\alpha_i A_i}(x_i) | \varphi_{\pi(i)} \rangle$$

more precise statement

need to consider in and out states

$|p_r, q_r, s_r, \sigma_r\rangle$

$p_r^2 = M_r^2$ other q-numbers spin

$|\{\theta_r\}\rangle_+$ / $|\{\theta_r\}\rangle_-$

in out

$|q_1 \dots q_n\rangle_+$ $|q_1 \dots q_n\rangle_-$

2) More precise

$$\lim_{t_{i_{min}} \to \infty} \langle 0 | T\left(O_1(x_1) \ldots O_n(x_n)\right) | \varphi_1 \ldots \varphi_n \rangle_{t_1 \ldots t_n}^{-}$$

$$= \sum_{\pi} \prod_i \langle 0 | O_i(x_i) | \varphi_{\pi(i)} \rangle$$

$$\lim_{t_{i_{max}} \to -\infty} {}_{+}\langle \varphi_1 \ldots \varphi_n | T\left(O_1(x_1) \ldots O_n(x_n)\right) | 0 \rangle$$

$$= \sum_{\pi} \prod_i \langle \varphi_{\pi(i)} | O_i(x_i) | 0 \rangle$$

• Similarly we can use $O_\chi(t) \equiv \int d^3x \, \mathcal{O}(t,x)\chi^*(t,x)$

Ex for scalars

$$\lim_{t_i \to \infty} \langle 0| T\left(O_{\chi_1}(t_1) \cdots O_{\chi_n}(t_n)\right) |q_1 \cdots q_n\rangle_-$$

$$= \sum_\pi \prod_i \langle 0| O_{\chi_i}(t_i) | q_{\pi(i)}\rangle = \sum_\pi \prod_i \langle \chi_i | q_{\pi(i)}\rangle$$

$$= \sum_\pi \prod_i \int \chi_i^*(p_i)\, q_{\pi(i)}(p_i)\, d\Omega_i$$

$$= \int \sum_\pi \prod_i \chi_i^*(p_{\pi(i)})\, d\Omega_{p_1} \cdots d\Omega_{p_n}\, q_1(p_1) \cdots q_n(p_n)$$

$$\equiv \underbrace{\lim_{\min t_i \to \infty} \langle 0 | T\left(O_{x_1}(t_1) \cdots O_{x_n}(t_n)\right) | p_1 \cdots p_n \rangle}_{\sum_{\pi} \prod_{i} \chi_i^*(p_{\pi(i)})} \times d\ell_1 \cdots d\ell_n \, \varphi_1(p_1) \cdots \varphi_n(p_n)$$

⚠ Lehmann-Symanzik-Zimmermann reduction formulae (LSZ)

• **Idea**: singularities of $G_N(p_1 \ldots p_N) \Longleftrightarrow$ S-matrix

• **Statement** (scalar operators $\mathcal{O}(x)$ & scalar particles $p^2 = m^2$, $s = 0$)

$$G(\overbrace{p_1 \ldots p_n}^{p_i^0 > 0} ; \overbrace{k_1, \ldots k_m}^{k_i^0 > 0}) = \prod_i^n \int d^4x_i \, e^{i p_i x_i} \cdot \prod_j^m d^4 y_j \, e^{-i k_j y_j} \times$$

$\sum p_i = \sum k_i$

$$\times \langle 0 | T\left(\mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \, \mathcal{O}(y_1) \ldots \mathcal{O}(y_m) \right) | 0 \rangle$$

$$\sim \prod_i^n \frac{i\sqrt{Z}}{p_i^2 - m^2 + i\varepsilon} \prod_j^m \frac{i\sqrt{Z}}{k_j^2 - m^2 + i\varepsilon} \langle p_1 \ldots p_n | S | k_1 \ldots k_m \rangle$$

where

- $\sqrt{Z} \equiv \langle 0 | \mathcal{O}(0) | p \rangle$

- $\sim$ holds when

$$p_i^0 \longrightarrow E_{p_i} \equiv \sqrt{\vec{p}_i^{\,2} + m^2}$$

$$k_j^0 \longrightarrow E_{k_j} \equiv \sqrt{\vec{k}_j^{\,2} + m^2}$$

Proof

- Strategy: hunt for singularities
- we shall find them for $x_i^0 \to +\infty$
  $y_i^0 \to -\infty$

in region $x_i^0 \to +\infty$, $y_i^0 \to -\infty$ $\Rightarrow$

$$\begin{cases} \langle 0| T\left(\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\,\mathcal{O}(y_1)\cdots\mathcal{O}(y_m)\right)|0\rangle \\ = \langle 0| T\left(\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\right) T\left(\mathcal{O}(y_1)\cdots\mathcal{O}(y_m)\right)|0\rangle \end{cases}$$

$$\mathbb{1} = \sum |\alpha\rangle\langle\alpha|$$

All states

$$\mathbb{1} = \mathbb{1}^2 = \sum_\alpha |\alpha_-\rangle\langle\alpha_-| \sum_\beta |\beta_+\rangle\langle\beta_+|)$$

$$\sum_{\alpha,\beta} \underbrace{|\alpha_-\rangle}\underbrace{\langle\alpha_-|\beta_+\rangle}\underbrace{\langle\beta_+|}$$

$$\sum_{\alpha,\beta} \langle 0| T(O(x_1)\cdots O(x_n)) |\alpha_-\rangle\langle\alpha_-|\beta_+\rangle\langle\beta_+| T(O(y_1)\cdots O(y_m))|0\rangle$$

- leading singularity $\Rightarrow$ $|\alpha\rangle_- = |q_1\cdots q_n\rangle_-$ $\quad {}_+\langle\beta| = {}_+\langle w_1\cdots w_m|$

$$\Rightarrow \langle 0| T(O(x_1)\cdots O(x_n))|q_1\cdots q_n\rangle_-\, {}_-\langle q_1\cdots q_n|w_1\cdots w_m\rangle_+\, {}_+\langle w_1\cdots w_m| T(O(y_1)\cdots O(y_m))|0\rangle \times$$

$$\times \frac{1}{n!\,m!}\int d\Omega_{q_1}\cdots d\Omega_{q_n}\, d\Omega_{w_1}\cdots d\Omega_{w_m}$$

use factorization

$\langle 0 | \mathcal{O}(x) | q \rangle$
$= \sqrt{Z}\, e^{-iqx}$

$$\lim_{x_{i\,min}^0 \to \infty} \langle 0 | T(\mathcal{O}(x_1) \cdots \mathcal{O}(x_n)) | q_1 \cdots q_n \rangle_-$$

$$= \sum_{\pi} \prod \langle 0 | \mathcal{O}(x_i) | q_{\pi_i} \rangle$$

$$= \sum_{\pi} \prod_i \sqrt{Z}\, e^{-i q_{\pi_i} \cdot x_i}$$

$$\lim_{y_{i\,min}^0 \to -\infty} {}_+\langle w_1 \cdots w_m | T(\mathcal{O}(y_1) \cdots \mathcal{O}(y_m)) | 0 \rangle$$

$$= \sum_{\pi} \prod_i \sqrt{Z}\, e^{i w_{\pi_i} \cdot y_i}$$

outgoing

$$\int d^4x_i \, e^{i p_i x_i} \cdot \sqrt{Z} \, e^{-i q_{\pi_i} x_i} \, d\Omega_{q_{\pi_i}}$$

$t_i \equiv x_i^0$ $\qquad x_i^0 > T_{min}$

$$\int dt_i \, d^3\vec{x}_i \, e^{i(p_i^0 - q_{\pi_i}^0) t_i - i(\vec{p}_i - \vec{q}_{\pi_i})\vec{x}_i} \, d\Omega_{q_{\pi_i}} \sqrt{Z}$$

$$\int dt_i \, (2\pi)^3 \delta^3(\vec{p}_i - \vec{q}_{\pi_i}) \, e^{i(p_i^0 - q_{\pi_i}^0) t_i} \, d\Omega_{q_{\pi_i}} \sqrt{Z}$$

$$\boxed{\int dt_i \, \frac{\sqrt{Z}}{2E_{p_i}} \, e^{i(p_i^0 - \sqrt{\vec{p}_i^{\,2} + m^2}) t_i}}$$

q

$q_{\pi_i}^0 = \sqrt{q_{\pi_i}^2 + m^2}$

$\Rightarrow \sqrt{p_i^2 + m^2} \equiv E_{p_i}$

$$\lim_{\varepsilon \to 0} \int_{T_{min}}^{\infty} dt_i \, \frac{\sqrt{Z}}{2E_{p_i}} \, e^{i(p_i^0 - E_{p_i})t_i - \varepsilon t_i}$$

$$= \frac{\sqrt{Z} \quad i \, e^{i(p_i^0 - E_{p_i})T_{min}}}{2E_{p_i}\left(p_i^0 - E_{p_i} + i\varepsilon\right)}$$

$$\simeq \frac{\sqrt{Z}}{2E_{p_i}} \frac{i}{\left(p_i^0 - E_{p_i} + i\varepsilon\right)} + \ldots$$

$$\simeq \frac{\sqrt{Z}}{\left(E_{p_i} + p_i^0\right)} \frac{i}{\left(p_i^0 - E_{p_i} + i\varepsilon\right)} + \ldots$$

$$\frac{\sqrt{Z}\, i}{\left(p_i^{0\,2} - E_{p_i}^2 + i\varepsilon\right)} + \dots \qquad E_{p_i}^2 = \vec{p}_i^{\,2} + m^2$$

$$= \frac{\sqrt{Z}\, i}{p_i^2 - m^2 + i\varepsilon} + \dots$$

incoming)

$$\int_{-\infty}^{T_{max}} dy_i^0 \int e^{-i K_i y_i}\, e^{i \omega_{\pi_i} y_i}\, d^3 y^i \, d\ell_{\omega_{\pi_i}}$$

$$\longrightarrow \frac{\sqrt{Z}\, i}{K_i^2 - m^2 + i\varepsilon}$$

- more rigour by folding with wave packet:

$$\int d^4x\, \mathcal{O}(x) e^{ipx} \longrightarrow \int \frac{d^3p}{(2\pi)^3} \varphi_i(\vec{p}) \int d^4x\, e^{ipx} \mathcal{O}(x) \equiv \int dt\, \mathcal{O}_x(t)$$

$$\Rightarrow \int_{t_i > t_+} d^4x_i\, e^{i(p_i - q_i)x_i} d\Omega_{q_i} \sqrt{Z} \longrightarrow \int \frac{d^3p_i}{(2\pi)^3} \varphi_i(\vec{p}_i) \int d^4x_i\, e^{i(p_i - q_i)x_i} d\Omega_{q_i} \sqrt{Z}$$

$$= \int d\Omega_{\underline{p}}\, \varphi_i(\underline{p}) \int_{t_+}^{\infty} e^{i(p_i^0 - E_p)t} \sqrt{Z}\, dt \approx \int \frac{d^3p}{(2\pi)^3} \frac{i\, \varphi_i(p) \sqrt{Z}}{2E_p(p_i^0 - E_p + i\varepsilon)}$$

$$\approx \int \frac{d^3 p}{(2\pi)^3} \frac{i\, q_i(\underline{p}) \sqrt{Z}}{(p_i^0)^2 - \vec{p}^{\,2} - m^2 + i\varepsilon}$$

- In synthesis

- for scalars $\langle p_1 \dots p_n | S | k_1 \dots k_m \rangle = \lim_{\substack{p_i^2 \to m^2 \\ k_j^2 \to m^2}} \prod_i \frac{p_i^2 - m^2}{i\sqrt{Z}} \prod_j \frac{k_j^2 - m^2}{i\sqrt{Z}} G(p_1 \dots p_n, k_1 \dots k_m)$

- $\sqrt{Z} \equiv \langle 0 | \mathcal{O}(0) | p \rangle \implies \frac{1}{\sqrt{Z}}$ factors

- for spinning particles $\sqrt{Z}\, u_{A\sigma}^{j_-j_+s}(p) = \langle 0 | O_A^{j_-j_+} | p, \sigma \rangle$

ideally we would need a "field" O_σ such that

$$\langle 0 | O_{\sigma'} | p, \sigma \rangle = \sqrt{Z}\, \delta_{\sigma'\sigma}$$