Multiparticle states & P^2 spectrum

$$|(p_1, r_1, \sigma_1); (p_2, r_2, \sigma_2); \cdots (p_n, r_n, \sigma_n)\rangle = |\psi_n\rangle$$

$$P^\mu|\psi_n\rangle = (p_1^\mu + \cdots p_n^\mu)|\psi_n\rangle$$

Simple Kinematics $\quad P^2 \geq (m_1 + \cdots m_n)^2$

- <u>Ex</u> single particle of mass m

$$|\psi_1\rangle \longrightarrow P^2 = m^2$$

$$|\psi_2\rangle \longrightarrow P^2 \geq 4m^2 \Longrightarrow 2 \text{ particle threshold}$$

$$|\psi_3\rangle \longrightarrow P^2 \geq 9m^2 \Longrightarrow 3 \quad " \quad "$$

- ~~Ex~~ two particles of mass m_1 et m_2

$|\Psi_1\rangle \longrightarrow$ either $P^2 = m_1^2$ or m_2^2

$|\Psi_2\rangle \Rightarrow P^2 \geq 4m_1^2$; $P^2 \geq 4m_2^2$ $P^2 \geq (m_1+m_2)^2$

etc.

⚠ "Ladder" operators (*Ex* real scalars)

$$O_{\chi_M}(t) = \int d^3x\, \mathcal{O}(t,x)\, \chi_M^*(t,x) \qquad \chi_M = \int \frac{d^3k}{(2\pi)^3} e^{-ikx} \chi(k)$$

$$k^\mu = \left(\sqrt{M^2 + \underline{k}^2}, \underline{k}\right)$$

$$O^\dagger_{\chi_M}(t) = \int d^3x\, \mathcal{O}(t,x)\, \chi_M(t,x)$$

- Assume $|p\rangle$: $P^\mu |p\rangle = p^\mu |p\rangle \qquad p^\mu p_\mu = m^2$

$$\langle p | O^\dagger_{\chi_M}(t) | 0\rangle = \int d^3x \frac{d^3k}{(2\pi)^3} e^{ipx - ikx} = \underbrace{\langle p | \mathcal{O}(0) | 0 \rangle}_{\sqrt{Z}} e^{i(E_m(p) - E_M(p))t} \chi(k)$$

$$|\psi\rangle = \int d\Omega_p\, \psi(p)\, |p\rangle$$

$$\langle\psi| O^{\dagger}_{\chi_M}(t)|0\rangle = \sqrt{Z}\int d\Omega_p\, \psi^*(p)\chi(p)\, e^{i(E_m(p)-E_M(p))t}$$

- $m = M \quad \Rightarrow \quad = \sqrt{Z}\int d\Omega_p\, \psi^*(p)\chi(p) = \sqrt{Z}\langle\psi|\chi\rangle$

- $m \neq M \xRightarrow{\text{Riemann-Lebesgue}} \lim\limits_{t\to\pm\infty} \langle\psi|O^{\dagger}_{\chi_M}(t)|0\rangle = 0$

- asymptotically $O^{\dagger}_{\chi_M}$ creates states of mass M

Position space picture

$$\langle\psi|O^{\dagger}_{\chi_M}(t)|0\rangle = \sqrt{Z}\int d\Omega_p \frac{d^3k}{(2\pi)^3} d^3x\, e^{-i(p-k)x+i(E_m(p)-E_M(k))t}\, \psi^*(p)\chi(k)$$

$$\sqrt{Z}\int d\Omega_p\, d\Omega_k\, 2E_M(k)\, e^{-ipx+iE_m(p)t}\psi^*(p)\, e^{ikx-iE_M(k)t}\chi(k)\, d^3x$$

$$= \sqrt{Z}\int d^3x\, \psi^*(x)\, 2i\frac{d}{dt}\chi(x)$$

- $$\langle 0|\, O^+_{\chi_M}(t)\,|\psi\rangle = \sqrt{Z}\int d\Omega_p\, \psi(p)\,\chi(p)\, e^{-i(E_M(p)+E_M(p))t}$$

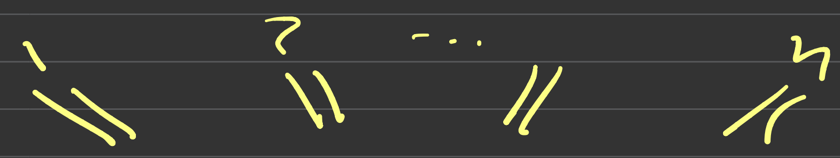
$$\forall\,|\psi\rangle \qquad \lim_{t\to\pm\infty}\ \langle 0|\, O^+_{\chi_M}(t))\,|\psi\rangle = 0$$

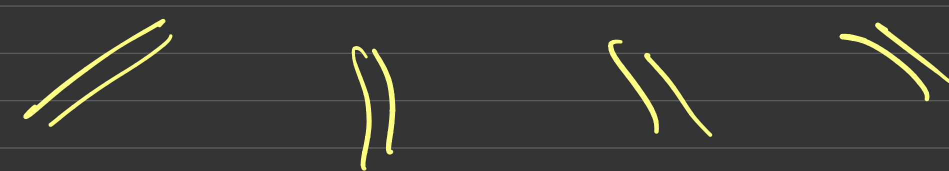
$$\Longrightarrow \quad \lim_{t\to\pm\infty}\ O_{\chi_M}(t)\,|0\rangle = 0$$

⚠ Consider now multiparticle states

$|q_1 \cdots q_n\rangle$ can be eiter $|q_1 \cdots q_n\rangle_-$ out

$|q_1 \cdots q_n\rangle_+$ in

$|q_1 \cdots q_n\rangle_- \sim$  $t \to \infty$

$|q_1 \cdots q_n\rangle \sim$  $t \to -\infty$

◎ Cluster Hp. (physically intuitive but derivable from Wightman axioms... ☺

$$\lim_{t\to\infty} \langle 0|T(O_{x_1}(t_1)\cdots O_{x_n}(t_n))|q_1 \cdots q_n\rangle_- = \sum_{\pi} \prod_{i=1}^{n} \langle 0|O_{x_i}(t_i)|q_{\pi(i)}\rangle$$

$$\overset{M=m}{=} \sum_{\pi} \prod_{i} \langle x_i | \varphi_{\pi(i)} \rangle$$

$$\lim_{t\to-\infty} {}_{+}\langle \varphi_1 \cdots \varphi_n | T(O^{+}_{x_1} \cdots O_{x_n}) | 0 \rangle = \sum_{\pi} \prod_{i=1}^{n} \langle \varphi_{\pi(i)} | O^{+}_{x_i}(t_i) | 0 \rangle$$

$$\overset{M=m}{=} \sum_{\pi} \prod_{i} \langle \varphi_{\pi(i)} | x_i \rangle$$

$\oplus$ $$\langle 0 | T(O_1 \cdots O_n) | \varphi_1 \cdots \varphi_m \rangle_{-} \xrightarrow[n\neq m]{t\to\infty} 0$$

$$ {}_{+}\langle \varphi_1 \cdots \varphi_m | T(O_1 \cdots O_n) | 0 \rangle \xrightarrow[t\to-\infty]{} 0$$

⊚ Statement straightforwardly generalized to

the case of non-identical particle and
to the case of particles with spin.

- Statement indeed holds true also removing either wave packets (on O or on $|\rangle$)

$$G(\overbrace{p_1 \dots p_n}^{p_i^0>0};\overbrace{k_1,\dots k_m}^{k_i^0>0}) = \prod_i^n \int d^4x_i\, e^{ip_i x_i} \cdot \prod_j^m d^4y_j\, e^{-ik_j y_j} \times$$

$$\Sigma p_i = \Sigma k_i$$

$$\times \langle 0| T\left(\mathcal{O}(x_1)\dots\mathcal{O}(x_n)\,\mathcal{O}(y_1)\dots\mathcal{O}(y_m)\right)|0\rangle$$

$$\sim \prod_i^n \frac{i\sqrt{Z}}{p_i^2-m^2+i\varepsilon} \prod_j^m \frac{i\sqrt{Z}}{k_j^2-m^2+i\varepsilon} \langle p_1 \dots p_n| S |k_1,\dots k_m\rangle$$

In order to make the computation well defined must introduce wave packets

$$\Rightarrow e^{ip_i x_i} \longrightarrow \chi_i^*(x_i) = \int \frac{d^3Q}{(2\pi)^3}\, e^{iE_i(Q)t_i - i\underline{Q}\cdot\underline{x}_i}\, \chi_i^*(Q)$$

$$E_i(Q) = \sqrt{p_i^2 + \underline{Q}^2} \qquad p_i^2 \equiv M_i^2$$

$$e^{-iK_a y_a} \longrightarrow \psi_a(y_a) = \int \frac{d^3 Q}{(2\pi)^3}\, e^{-iE_a(Q)T_a + iQ\cdot y_a}\, \psi_a(Q)$$

$$E_a(Q) = \sqrt{K_a^2 + \underline{Q}^2}$$

$$\Rightarrow \int \prod_i dt_i \prod_a dT_a \, \langle 0 | T\left(O_{x_1}(t_1) \cdots O_{x_n}(t_n)\, O^\dagger_{\psi_1}(T_1) \cdots O^\dagger_{\psi_m}(T_m)\right) | 0 \rangle$$

$\Rightarrow$ • each dt_i, dT_a can in principle give a divergence from the asymptotic $t \to \pm\infty$ regions

$$\int dt \sim \frac{1}{\Delta E}$$

- The asymptotic cluster formula is easily seen to imply such divergences can (<u>only</u>) arise in the region $t_i \to +\infty$ $T_e \to -\infty$

- In these region:

- $\langle 0 | T(O_{x_1} \cdots O_{x_n} O^\dagger_{t_1} \cdots O^\dagger_{t_m}) | 0 \rangle = \langle T(O_{x_1} \cdots O_{x_n}) \, T(O^\dagger_{t_1} \cdots O^\dagger_{t_m}) | 0 \rangle$

- we insert $\mathbb{1}$ between the two operator strings

write $\mathbb{1} = \mathbb{1} \cdot \mathbb{1} = \displaystyle\sum\!\!\!\!\!\!\int |\alpha_-\rangle\langle\alpha_-| \cdot \sum\!\!\!\!\!\!\int |\beta_+\rangle\langle\beta_+|$

$$= \sum\!\!\!\!\!\!\int |\alpha_-\rangle \langle\alpha_- | \beta_+\rangle \langle \beta_+|$$

$$= \sum\!\!\!\!\!\!\int |\alpha_-\rangle S_{\alpha\beta} \langle \beta_+|$$

• our expression then becomes

$$\int \!\!\sum\!\!\!\!\!\!\int \langle 0| T\left(O_{x_1}(t_1) \cdots O_{x_n}(t_n)\right) |\alpha_-\rangle \langle \alpha|\beta\rangle_+ \, {}_+\langle \beta| T\left(O^+_{y_1}(T_1) \cdots O^+_{y_m}(T_m)\right)|0\rangle$$

$$\prod dt_i \prod dT_a$$

$\Rightarrow$ leading singularity $|\alpha_-\rangle = |q_1 \cdots q_n\rangle_-$ $\quad {}_+\langle\beta| = {}_+\langle w_1 \cdots w_m|$

$$= \int dt_i \, dT_a \, \langle 0|T\left(O_{x_1} \cdots O_{x_n}\right)|q_1 \cdots q_n\rangle_- \, {}_-\langle q_1 \cdots q_n| w_1 \cdots w_m\rangle_+ \, {}_+\langle w_1 \cdots w_m| T\left(O^+_{y_1} \cdots O^+_{y_m}\right)|0\rangle$$

$$\frac{1}{n!\,m!} \prod_i^n d\Omega_{q_i} \prod_a^m d\Omega_{w_a}$$

- To estimate the contribution of the asymptotic regions we can now use the factorization hypothesis

- Consider <u>out</u> contribution first

$$\lim_{\min\{t_i\}\to\infty} \langle 0| \mathcal{O}_{x_1}(t_1)\cdots\mathcal{O}_{x_n}(t_n)|q_1\cdots q_n\rangle = \sum_{\pi}\prod_i \langle 0|\mathcal{O}_{x_i}(t_i)|q_{\pi(i)}\rangle$$

$\downarrow$

in reality for $t_i \gg T_*$ $\leftarrow$ determined by wave packets

- each factor gives

$$\int_{T_*}^{\infty} dt_i \,\langle 0|\mathcal{O}_{x_i}(t_i)|q_{\pi(i)}\rangle = \sqrt{Z}\int_{T_*}^{\infty} dt_i \, e^{i(E_{M_i}(q_{\pi_i})-E_m(q_{\pi_i}))t_i}\, x_i^*(q_{\pi_i})$$

$$M_i^2 \equiv P_i^2$$

where we recalled

$$\boxed{\begin{aligned}\langle \omega | \mathcal{O}^{\dagger}_{\chi_\pi}(t)|0\rangle &= \sqrt{Z}\ \ \chi(\omega)\, e^{i(E_m(\omega)-E_\pi(\omega))t}\\ \langle 0| \mathcal{O}_{\chi_\pi}(t)|\omega\rangle &= \sqrt{Z}\ \ \chi^{\dagger}(\underline{\omega})\, e^{i(E_\pi(\underline{\omega})-E_m(\underline{\omega}))t}\end{aligned}}$$

- for $M_i \neq m$ integral converges thanks to smearing from $\int d\Omega_{q_{\pi(i)}} \Rightarrow$ ok to add convergence factor $e^{-\varepsilon t_i}$

$$\int_{T_*}^{\infty} dt_i\, e^{i(E_{M_i}(q_{\pi_i})-E_m(q_{\pi_i})+i\varepsilon)t_i} = \frac{i}{E_{M_i}(q_{\pi_i})-E_m(q_{\pi_i})+i\varepsilon}$$

Combined with $d\Omega_{q_{\pi(i)}}$ each factor then gives

$$\frac{i\sqrt{Z}}{2E_m(q_{\pi_i})\left(E_{\sqrt{P_i^2}}(q_{\pi_i})-E_m(q_{\pi_i})+i\varepsilon\right)}\quad \frac{d^3 q_{\pi_i}}{(2\pi)^3}\quad \chi_i^*(q_{\pi_i})$$

$$\simeq \frac{i\sqrt{Z}\;\chi_i^*(q_{\pi_i})}{\left(E_{\sqrt{p_i^2}}(q_{\pi_i}) + E_m(q_{\pi_i})\right)\left(E_{\sqrt{p_i^2}}(q_{\pi_i}) - E_m(q_{\pi_i}) + i\varepsilon\right)} \frac{d^3 q_{\pi_i}}{(2\pi)^3}$$

$$= \frac{i\sqrt{Z}}{p_i^2 - m^2 + i\varepsilon}\, \chi_i^*(q_{\pi_i}) \frac{d^3 q_{\pi_i}}{(2\pi)^3}$$

Now

$$\frac{1}{n!}\sum_{\pi} \prod_i \int \langle 0 | O_{\chi_i} | q_{\pi(i)} \rangle \, d\Sigma_{q_{\pi(i)}}\, dt_i \cdot \underbrace{\langle q_1 \cdots q_n |}_{\text{symmetric}} \cdots$$

$$= \prod_{i=1}^{n} \frac{i\sqrt{Z}}{p_i^2 - m^2 + i\varepsilon}\, \chi_i^*(q_i) \frac{d^3 q_i}{(2\pi)^3} \langle q_1 \cdots q_n |$$

- A similar procedure applies for the <u>in</u> factor

In that case factors are

$$\int_{-\infty}^{-T_*} \langle \omega_{\pi_a} | O^\dagger_{+_a}(T_a) | 0 \rangle \, dT_a \, d\ell_{\omega_{\pi_a}} =$$

$$= \frac{d^3 \omega_{\pi_a}}{(2\pi)^3} \frac{i\sqrt{2}}{K_a^2 - m^2 + i\varepsilon} \psi_a(\underline{\omega}_{\pi_a})$$

Thus in the end we have

$$\prod_i^n \int \frac{d^3 \underline{p}_i}{(2\pi)^3} \chi_i^*(\underline{p}_i) \prod_a^m \int \frac{d^3 k_a}{(2\pi)^3} \psi_a(\underline{k}_a) \, G(p_1, \dots p_n; k_1, \dots k_m)$$

$$p_i^0 = \sqrt{\underline{p}_i^2 + \vec{p}_i^{\,2}} \qquad k_a^0 = \sqrt{\underline{k}_a^2 + \vec{k}_i^{\,2}}$$

$$\simeq \prod_i^n \int \frac{d^3 p_i}{(2\pi)^3} \chi_i^*(p_i) \prod_a \int \frac{d^3 k_a}{(2\pi)^3} \psi_a(k_a) \qquad \times$$

$$\times \prod_{i=1}^n \frac{\sqrt{z}\, i}{\underline{p}_i^2 - m^2 + i\varepsilon} \prod_a^m \frac{\sqrt{z}\, i}{\underline{k}_a^2 - m^2 + i\varepsilon} \, {}_-\langle p_1 \dots p_n | k_1 \dots k_m \rangle_+$$

⚠ Consider instead

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$$\langle \chi | \mathcal{O}_\chi(t) | 0 \rangle = \iint d\Omega_k \, d\Omega_p \, \varphi^*(k) \langle k | \mathcal{O}(t,x) | 0 \rangle \, 2E_p \, e^{ipx} \chi^*(p)$$

$$= \int d\Omega_k \, d\Omega_p \, e^{i(p+k)x} \, 2E_p \, \varphi^*(k) \chi^*(p) \sqrt{Z}$$

$$= \int d\Omega_k \, e^{2iE_k t} \varphi^*(k) \chi^*(-k) \sqrt{Z}$$

$\longrightarrow 0$ for $t \longrightarrow \pm\infty$

Parity, wave functions & projectors

- Back to the basics

$$\left(\tfrac{1}{2},0\right) \longrightarrow (\psi_L)_\alpha$$

$$\left(0,\tfrac{1}{2}\right) \longrightarrow (\psi_R)^{\dot\beta}$$

Parity

- from QFT1-2 lecture notes

$$(\psi_R{}^{\dot\beta})^* = \psi_L{}^{\beta} \qquad (\psi_{L\alpha})^* = \psi_{R\dot\alpha}$$

- $$U_P^\dagger(\psi_{L\alpha})U_P = P_{\alpha\dot\beta}\,\psi_R{}^{\dot\beta} \qquad P_{\alpha\dot\beta} = \sigma^0{}_{\alpha\dot\beta}$$

$$U_P^\dagger(\psi_R{}^{\dot\beta})U_P = P^{\dot\beta\alpha}\,\psi_{L\alpha} \qquad P^{\dot\beta\alpha} = \bar\sigma^{0\,\dot\beta\alpha}$$

- j_-, j_+ $\quad \mathcal{O}_{\underbrace{\alpha_1 \dots \alpha_{j_-}}_{\text{symm}}}^{\overbrace{\dot\beta_1 \dots \dot\beta_{j_+}}^{\text{symm}}} \equiv \mathcal{O}^{L}{}_{A}{}^{\dot B} \equiv \mathcal{O}^{L}_{\mathcal{A}}$

- j_+, j_- $\quad \mathcal{O}^{\dot\alpha_1 \dots \dot\alpha_{j_-}}{}_{\beta_1 \dots \beta_{j_+}} \equiv \mathcal{O}^{R\dot A}{}_{B} \equiv \mathcal{O}^{R\dot{\mathcal{A}}}$

- $\left(\mathcal{O}^{R\dot A}{}_{B}\right)^* = \mathcal{O}^{LA}{}_{\dot B}$

- $\text{Inv} = \mathcal{O}^{LA}{}_{\dot B} \cdot \mathcal{O}^{L}{}_{A}{}^{\dot B} \equiv \mathcal{O}^{L\mathcal{A}} \mathcal{O}^{L}_{\mathcal{A}}$

Wave functions

- $\psi^{(j_-,j_+)s}_{\mathcal{A}\sigma} \equiv \psi^{L,s}_{\mathcal{A}\sigma}(p) = \langle 0 | \mathcal{O}^{L}_{\mathcal{A}}(0) | p, \sigma \rangle$

$$= D_L(H(p))_A{}^B \, \underset{\substack{\downarrow \\ \text{suitably} \\ \text{normalized}}}{T^{L,s}_{B\sigma}} \, \underset{\downarrow}{\sqrt{Z_L}}$$

- $(\psi^{j_+, j_-}{}_{\sigma}{}^{\dot{A}}) \equiv \psi^{R\dot{A}}{}_{\sigma}(p) = \langle 0 | O^{R\dot{A}}(0) | p, \sigma \rangle$

$$= D_R(H(p))^{\dot{A}}{}_{\dot{B}} \, T^{R,s\,\dot{B}}{}_{\sigma} \, \sqrt{Z_R}$$

⚠ Comments

- $T^{L,s}_{B\sigma}$ and $T^{R,s\,\dot{B}}{}_{\sigma}$ commute with $SO(3)$ (little group)

$$T^{L,s}_{B\sigma} = \langle 0 | U(R) U(R)^{\dagger} O^{L}_{B} U(R) U(R)^{\dagger} | \bar{p}, \sigma \rangle \frac{1}{\sqrt{Z_L}}$$

$$= D^L(R)_B{}^A \, T^{L,S}_{A\sigma'} \, D^S_{\sigma'\sigma}(R^{-1})$$

$$D^L_A{}^B \, T^{L,S}_{B\sigma} = T^{L,S}_{A\sigma'} \, D_{\sigma'\sigma}(R)$$

by Schur $T_{A\sigma}$ ~ projector on spin S

- In parity invariant theory

$$U_P^\dagger \, O^L_A \, U_P = P_{A\dot{B}} \, O^{R\,\dot{B}} \Rightarrow T^{L,S}_{AS} = P_{A\dot{B}} \left(T^{R,S}\right)^{\dot{B}}$$

Based on R-invariance we now have

$$\sum_A \left(T^{R,S\,\dot{A}}{}_{\sigma}\right)^* \left(T^{L,S}{}_{A\sigma'}\right) = C \, \delta_{\sigma\sigma'}$$

$\Rightarrow$ pick normalization such that $C = 1$

$$\Rightarrow \sum_\sigma \left(T^{L,s}{}_{A\sigma}\right)\left(T^{R,s\,\dot B}{}_{\sigma}\right)^* = \Pi^{s}{}_{A}{}^{B}$$

$\equiv$ projector on spin s subspace

$$\left(\delta_A{}^B - \Pi^{s}{}_{A}{}^{B}\right) T^{L,s}{}_{B\sigma} = 0$$

wave equation in rest frame

- Boosting
  - $\psi^{L,s}{}_{B,\sigma}(p) = D^L(H(p))_A{}^B\, T^{L,s}{}_{B\sigma}$
  - $\Pi_A{}^B(p) = \psi^{L,s}{}_{A,\sigma}(p)\left(\psi^{R,s\,\dot B}{}_{\sigma}(p)\right)^*$

    $= D^L(H(p))\,\Pi(0)\,D^{R\,+}(H(p))$

$$= D^{L}(H(p))\, \Pi(0)\, D^{L^{-1}}(H(p))$$

$$\Rightarrow \boxed{\left(\delta^{A}_{B} - \Pi^{s}_{B}{}^{A}(p)\right)\psi^{L,S}_{A\sigma}(p) = 0}$$

- In synthesis

- for scalars $\langle p_1 \dots p_n | S | k_1 \dots k_m \rangle = \lim_{\substack{p_i^2 \to m^2 \\ k_j^2 \to m^2}} \prod_i \frac{p_i^2 - m^2}{i\sqrt{Z}} \prod_j \frac{k_j^2 - m^2}{i\sqrt{Z}} G(p_1 \dots p_n, k_1 \dots k_m)$

- $\sqrt{Z} \equiv \langle 0 | \mathcal{O}(0) | p \rangle \Rightarrow \frac{1}{\sqrt{Z}}$ factors

- for spinning particles $\sqrt{Z}\, \psi^{j, j_3, s}_{A\sigma}(p) = \langle 0 | \mathcal{O}^{j, j_3}_{A} | p, \sigma \rangle$

ideally we would need a "field" $\mathcal{O}_\sigma$ such that

$$\langle 0 | \mathcal{O}_{\sigma'} | p, \sigma \rangle = \sqrt{Z}\, \delta_{\sigma'\sigma}$$

- Contraction with the parity conjugated wave function does precisely that

$$\underline{\underline{\text{def}}} \quad O_\sigma(p,x) = \left(\psi^{R\dot{A}}_{\sigma}(p)\right)^* O_A(x) = \left[\bar{\psi}^R(p)^*\right]^A_\sigma O_A(x)$$

$$\langle 0|\, O_\sigma(p,x)\,|p,\sigma'\rangle = \psi^{R\dagger}_\sigma \psi^L_{\sigma'} \sqrt{Z} = \delta_{\sigma\sigma'}\sqrt{Z}$$

- wave packets

$$\int O_A(x) \left(\psi^{R*}\right)^A_\sigma(p)\, \chi^*_\sigma(p)\, e^{+ipx}\, d^3x$$

$$= O_\chi(t)$$