Overview

I. Representations of Poincaré group

II. LSZ approach to S matrix

III. Path integral approach to QFT:
Functional methods, Ward identities, ...

IV. Renormalization

V. Quantized gauge theories

VI. Infrared effects

In QFT 1 & 2 :

1. Realize Poincaré via local field theory (fields & Lagrangian)
2. Canonical Quantization

$$\Downarrow$$

A. Hilbert space $\equiv$ Fock space $\Longrightarrow$ particles
B. Noether charges generate unitary representation of Poincaré
C. Single particle states $\equiv$ irreps

<u>Here</u>: Q.M. ⊕ Relativity

I. Poincaré in Hilbert space: irreps of $ISO(3,1)$

$\Longrightarrow$ one particle states (mass, spin, helicity)

II. Poincaré on quantum fields (of arbitrary spin)

$\Longrightarrow$ wave functions and wave eqs

III. "Axiomatic" approach to S-matrix: LSZ

$ISO(3,1)$: $\quad x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu} \quad\quad \Lambda \in SO(3,1)$

$$g \equiv g(\Lambda, a) \quad \begin{cases} g(\Lambda_1, a_1)\, g(\Lambda_2, a_2) = g(\Lambda_1 \Lambda_2, \Lambda_1 a_2 + a_1) \\ g^{-1}(\Lambda, a) = g(\Lambda^{-1}, -\Lambda^{-1} a) \\ g(\Lambda, a) = g(\mathbb{1}, a)\, g(\Lambda, 0) \end{cases}$$

Generators & Exponential Map

$$U(g(\Lambda, a)) \equiv U(\Lambda, a)$$

$$U(g(\Lambda, 0)) \equiv U(\Lambda) = \exp\left(\frac{i\omega_{\mu\nu}}{2} J^{\mu\nu}\right)$$

$$U(g(\mathbb{1}, a)) \equiv U(a) = \exp\left(i a_{\mu} P^{\mu}\right)$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(\eta^{\nu\rho} J^{\mu\sigma} + \eta^{\mu\sigma} J^{\nu\rho} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho}\right)$$

$$[P^{\mu}, J^{\rho\sigma}] = i\left(\eta^{\mu\rho} P^{\sigma} - \eta^{\mu\sigma} P^{\rho}\right)$$

$$[P^{\mu}, P^{\nu}] = 0$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(\eta^{\mu\rho} J^{\sigma\nu} - \eta^{\mu\sigma} J^{\rho\nu} + \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\nu\sigma} J^{\mu\rho}\right)$$

$$J^{i} \equiv \frac{1}{2} \varepsilon^{ijk} J^{jk} \qquad K^{i} \equiv J^{i0}$$

$$g(\Lambda^{-1},0)\,g(\mathbb{1},a)\,g(\Lambda,0) = g(\Lambda^{-1},0)\,g(\Lambda,a) = g(\mathbb{1},\Lambda^{-1}a)$$

$$U(g(\Lambda,0)) \equiv U(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}$$

$$U(g(\mathbb{1},a)) \equiv U(a) = e^{iP^{\mu}a_{\mu}}$$

or $U(\Lambda^{-1})P^{\mu}U(\Lambda) = \Lambda^{\mu}{}_{\nu}P^{\nu}$

$$U(\Lambda^{-1})U(a)U(\Lambda) = U(\Lambda^{-1}a) = e^{iP^{\mu}(\Lambda^{-1}{}_{\mu}{}^{\nu}a_{\nu})}$$

$$= e^{i(\Lambda^{\nu}{}_{\mu}P^{\mu})a_{\nu}}$$

$$U(\Lambda)^{-1}\,e^{iP^{\nu}a_{\nu}}\,U(\Lambda) =$$

$$= e^{i U^{-1}(\Lambda) P^{\nu} U(\Lambda) a_{\nu}} = e^{i (\Lambda^{\nu}{}_{\mu} P^{\mu}) a_{\nu}}$$

$$\boxed{U^{-1}(\Lambda) P^{\nu} U(\Lambda) = \Lambda^{\nu}{}_{\mu} P^{\mu}}$$

while by checking $[J, J]$ commutator we get

$$U(\Lambda^{-1}) J^{\mu\nu} U(\Lambda) = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} J^{\mu'\nu'}$$

- $ISO(3,1)$ non-compact $\Longrightarrow$ unitary reps infinite dim
  - Hilbert space $\equiv \bigoplus$ irreps
  - Find all unitary irreps ($J^{\mu\nu}$, $P^\nu \equiv$ hermitean)

- Question: how many and which quantum numbers label the irreps?

Casimirs $\equiv C$

- $U(\Lambda)^{-1} P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu$ $\qquad$ $U(\Lambda)^{-1} J^{\mu\nu} U(\Lambda) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma J^{\rho\sigma}$

- $[J_{\mu\nu}, C] = 0 \implies$ C built from J, P by contracting all indices with $\eta_{\mu\nu}, \epsilon_{\mu\nu\rho\sigma}$

- $[P_\mu, C] = 0$ $\qquad$ less obvious

Candidates

- $J_{\mu\nu} J^{\mu\nu}$, $P_\mu P^\mu$

- $J_{\mu\nu} J^\nu{}_\rho J^{\rho\mu}$, $J_{\mu\nu} P^\mu P^\nu$, ...

- $ISO(3,1)$ invariants

- $[X, X] \sim X \implies \underbrace{X \dots X}_{n}$ ordering inessential up to $X^{n-1}, X^{n-2}, \dots$

- $SO(3,1)$ invariants from $J^{\mu\nu}, P^\mu$ | can treat them as c-numbers

Up to $SO(3,1)$

$$J^{\mu\nu} = \left(\begin{array}{cc|cc} 0 & E & & \\ -E & 0 & & 0 \\ \hline & & 0 & B \\ & 0 & -B & 0 \end{array}\right) \qquad P^\mu = \begin{pmatrix} P \\ 0 \\ P' \\ 0 \end{pmatrix}$$

$\Longrightarrow$ 4 invariants

Ex

$$J^{\mu\nu}\,\delta_{\mu\nu} \qquad J^{\mu\nu}\,J^{\rho\sigma}\,\epsilon_{\mu\nu\rho\sigma}$$

$$W_\mu\,W^\mu \qquad P^\mu P_\mu$$

$\downarrow$

$$W^\mu \equiv \frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}\,J_{\nu\rho}\,P_\sigma \qquad [W^\mu, P^\nu] = 0$$

• $[W^\mu, P_\nu] = 0 \implies [W^\mu W_\mu, P_\nu] = 0$ $\underline{\text{ok}}$

$$\boxed{\begin{array}{ll} W^2 = W^\mu W_\mu & \text{are the two Casimirs of } ISO(3,1) \\ P^2 = P^\mu P_\mu & \end{array}}$$

As we know $P^2 = M^2 \longrightarrow$ mass

$W^2 = -J(J+1)M^2 \longrightarrow$ spin

(or helicity at $M=0$)

- $P^\mu P_\mu \longrightarrow M^2$ <u>mass</u>

- $W^\mu W_\mu \longrightarrow$ · compute in C.M. where $P^\mu = (M, 0)$

  · $W^\mu = M(0, -\vec{J}) \Rightarrow W^2 = -M^2 J_i J^i$

  $= -M^2 s(s+1)$

<u>Spin</u>

Construction of Unitary irreps

$$U(\Lambda, a) \equiv e^{iP\cdot a} e^{-\frac{i}{2}\omega\cdot J} \qquad P_\mu, J_{\mu\nu} \equiv \text{hermitean}$$

$$U(\Lambda) \equiv e^{-\frac{i}{2}\omega\cdot J}$$

$[P_\mu, P_\nu] = 0 \implies$ simultaneously diagonalized

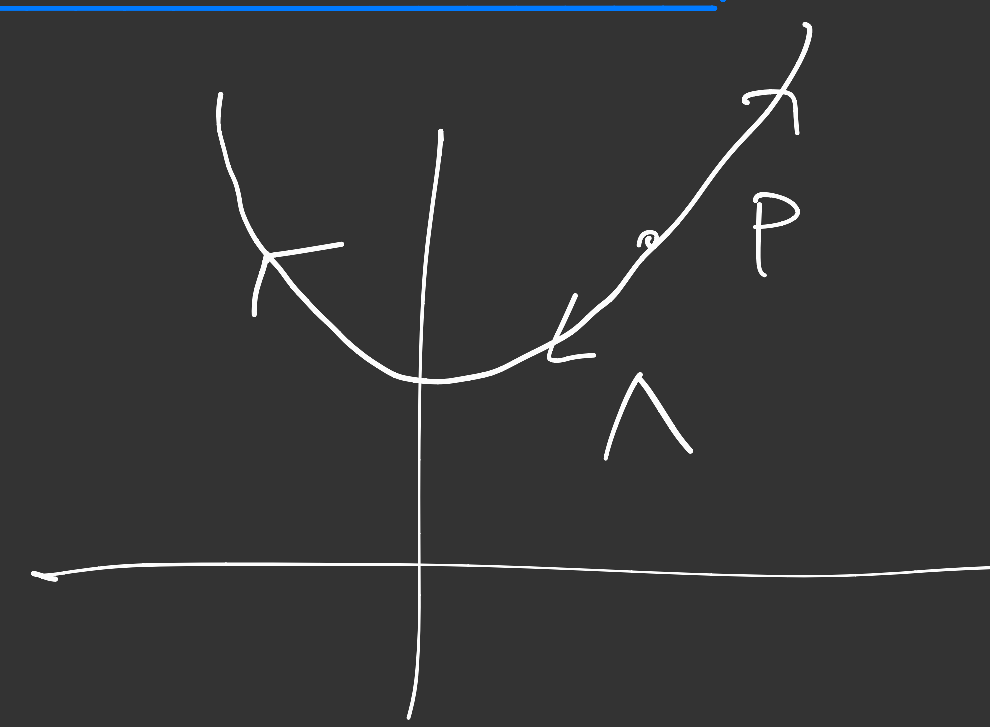
$$|\psi\rangle = |\underbrace{p}_{p^0 \dots p^3}, \underbrace{\sigma}_{\text{all other q-numbers}}\rangle$$

$$P^\mu |\psi\rangle = p^\mu |\psi\rangle$$

$$P^\mu U(\Lambda)|p,\sigma\rangle = U(\Lambda)U(\Lambda)^\dagger P^\mu U(\Lambda)|p,\sigma\rangle$$

$$= U(\Lambda)\Lambda^\mu{}_\nu P^\nu|p,\sigma\rangle = \left(\Lambda^\mu{}_\nu p^\nu\right)U(\Lambda)|p,\sigma\rangle$$

$$\Rightarrow \boxed{U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'}|\Lambda p,\sigma'\rangle C_{\sigma'\sigma}(\Lambda,p)}$$



- given $|p,\sigma\rangle$ can always bring p to some reference $\bar{p}$ via Lorentz transformation

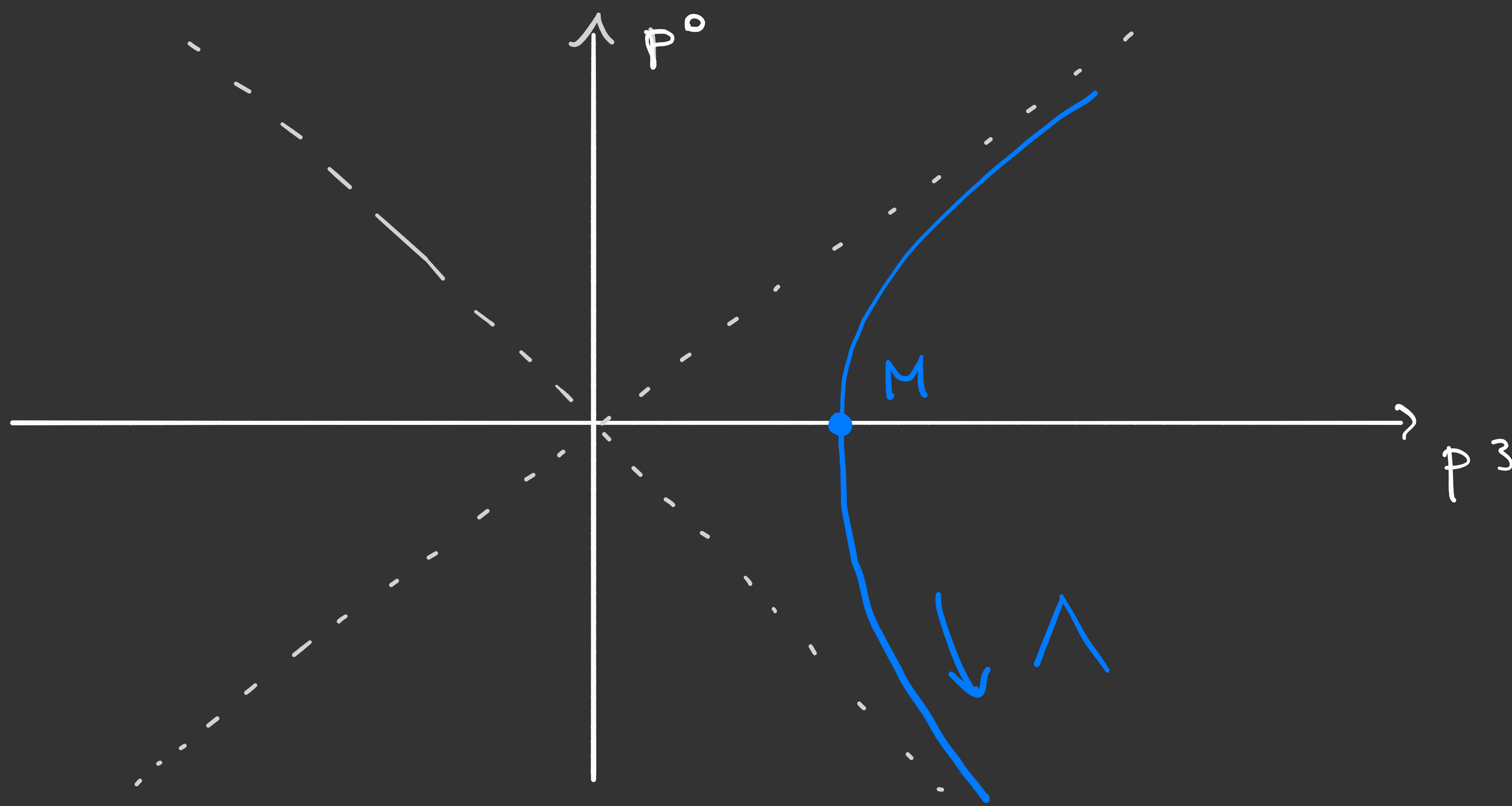
- possibilities for $\bar{p}$ depend on
  - $p^\mu p_\mu \equiv p^2$
  - $\text{sign}(p^0)$

$$\boxed{\begin{array}{lll} \text{I. } p^2 = M^2 > 0 & \Longrightarrow & \bar{p} = (M, \underline{0}) \\ \text{II. } p^2 = 0 & \Longrightarrow & \begin{cases} \text{II}_A: \ \bar{p} = (K, 0, 0, K) \\ \text{II}_B: \ \bar{p} = (0, 0, 0, 0) \end{cases} \\ \text{III. } p^2 = -M^2 < 0 & \Longrightarrow & \bar{p} = (0, 0, 0, M) \end{array}}$$

$M > 0$

$K > 0$

However, case III implies states with $p^0 < 0$



III $\quad P^0 \equiv E$ not positive $\Longrightarrow$ unphysical

△ <u>Little group</u>: subgroup $G_L \subset SO(3,1)$ that leaves $\bar{p}$ invariant

- $\forall \Lambda_L \in G_L$, $U(\Lambda_L)|\bar{p},\sigma\rangle = \sum_{\sigma'} |\bar{p},\sigma'\rangle C_{\sigma'\sigma}(\Lambda_L)$

$\Rightarrow \{|\bar{p},\sigma\rangle\}$ supports a representation of G_L

		G_L
I.	$\bar{p}^\mu = (M,0,0,0)$	$SO(3)$
II_A	$\bar{p}^\mu = (K,0,0,K)$	$ISO(2)$
II_B	$\bar{p}^\mu = (0,0,0,0)$	$SO(3,1)$
III	$\bar{p}^\mu = (0,0,0,M)$	$SO(2,1)$

Except for $\mathbb{I}_B$ G_L generated by $W^\mu(\bar{P}) = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho}\bar{P}_\sigma$

Indeed

- $[W_\mu, P_\nu] = 0 \Rightarrow W_\mu$ acts <u>invariantly</u> on eigenspaces of P^μ

- $$\boxed{P^\mu|\psi\rangle = k^\mu|\psi\rangle \Longrightarrow P^\mu W^\nu|\psi\rangle = W^\nu P^\mu|\psi\rangle = k^\mu[W^\nu|\psi\rangle]}$$

- $$\boxed{W^\mu|\psi\rangle = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma|\psi\rangle = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}|\psi\rangle k_\sigma}$$

- $$\boxed{\underline{\underline{\text{Ex}}} \quad k = (M, 0) \Longrightarrow W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho 0}J_{\nu\rho}M = (0, -M\vec{J})}$$

$\triangle$ Method : Induced Representation

- construct irrep of G_L over $|\bar{p}, \sigma\rangle$
- construct all other states by boosting $U(\Lambda)|\bar{p}, \sigma\rangle$

II_B) Simplest case : $|0, \sigma\rangle$

$G_L = SO(3,1) \equiv$ non-compact $\to$ infinite

$\to$ trivial $\equiv$ 1-dim

I) $P^2 = M^2$

$\bar{P} = (M, \underline{0})$

$|\bar{P}, \sigma\rangle$ $\quad G_L \equiv SO(3)$ generated by J^i

$$W^i \sim \frac{1}{2} \varepsilon^{ijk0} J_{jk} M \sim J^i$$

$\underline{\underline{|\bar{P}, \sigma\rangle}} \equiv$ representation of $SO(3)$

- pick irreducible $|\bar{p}, \sigma\rangle$
- options labelled by $J_i J_i = s(s+1)$

  $s =$ half-integer

  dimension $= 2s+1$

$\Longrightarrow$ pick a certain s

- choose standard basis of J_3 eigenvalues

$$
\begin{array}{ccccc}
|\bar{p}, -s\rangle, & |\bar{p}, -s+1\rangle, & \dots & , & |\bar{p}, s\rangle \\
\downarrow & \downarrow & & & \downarrow \\
J_3 = -s & J_3 = -s+1 & & & J_3 = s
\end{array}
$$

$$\boxed{\sigma \equiv J_3 = m}$$

- $P^\mu = (M, 0) \longrightarrow P^\mu P_\mu = M^2$

- $W^\mu = (0, -M\vec{J}) \longrightarrow W^\mu W_\mu = -M^2 \vec{J}\cdot\vec{J}$

$$= -M^2 s(s+1)$$

$$\boxed{\begin{gathered} P^2 = M^2 \\ W^2 = -M^2 s(s+1) \end{gathered}}$$

$$U(\Lambda)\,|p,\sigma\rangle = |\Lambda p, \sigma'\rangle\, C_{\sigma'\sigma}(\Lambda, p)$$

- $U(\Lambda)|\bar{p},\sigma\rangle = \sum_{\sigma'} |\Lambda\bar{p},\sigma'\rangle C_{\sigma'\sigma}(\Lambda,\bar{p})$

- $\forall\, p \mid p \in \{\Lambda\bar{p}$ hyperboloid$\}$, choose

  a specific Λ_p such that $p = \Lambda_p \bar{p}$

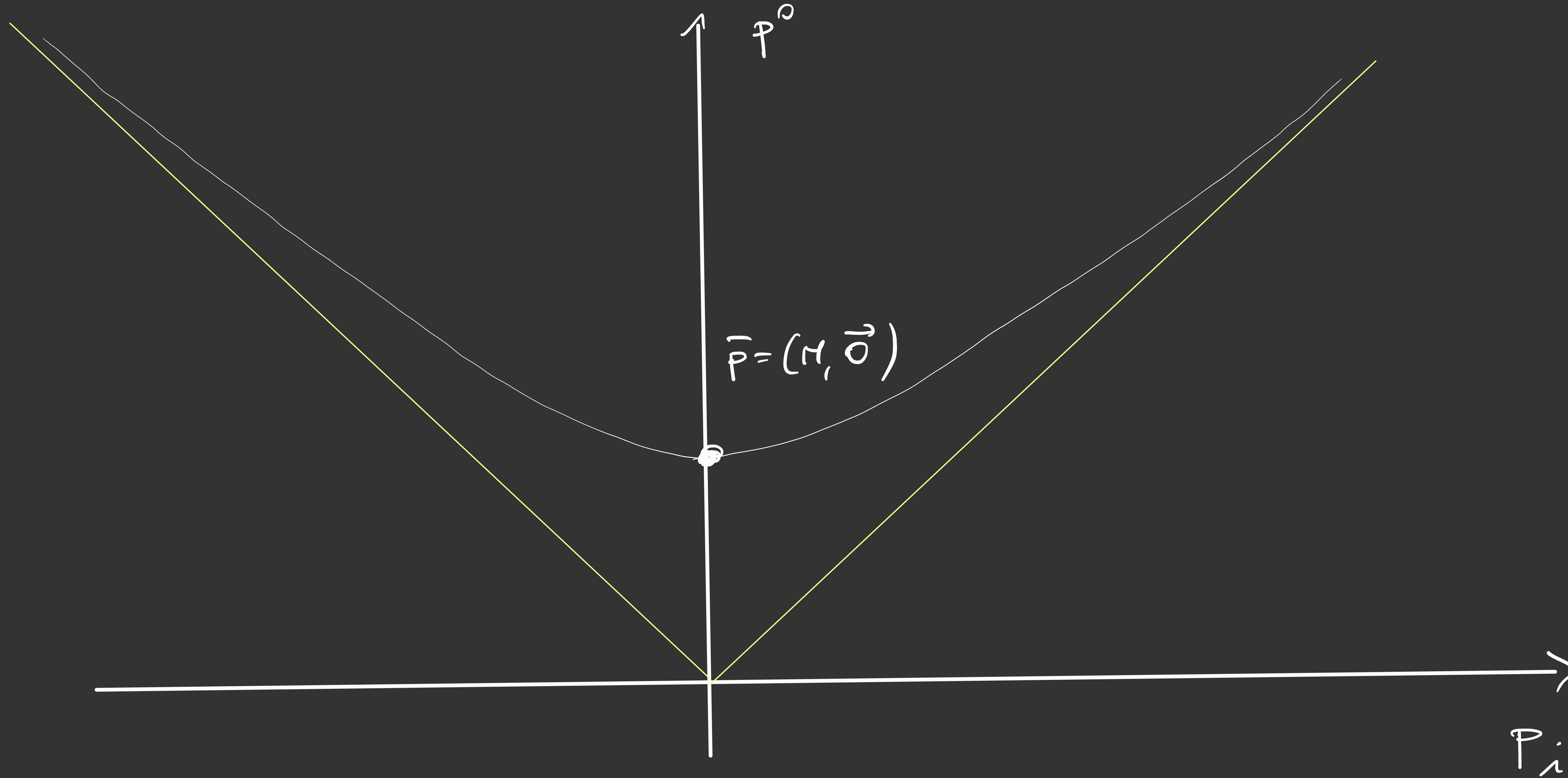
- choice not unique:

$$\Lambda'_p \equiv \Lambda_p R$$

$$\Lambda'_p \bar{p} = \Lambda_p R \bar{p} = \Lambda_p \bar{p} = p$$

- define $|p,\sigma\rangle \equiv U(\Lambda_p)|\bar{p},\sigma\rangle$

$$H(p) \equiv U(\Lambda_p)$$


p^0
$\bar{p} = (M, \vec{0})$
p_i

• Action of $U(\Lambda, a)$ on $\{|p,\sigma\rangle\}$ fully determined

• Translations

$$e^{iP\cdot a}|k,\sigma\rangle = e^{iP\cdot a} H(k)|\bar{p},\sigma\rangle =$$

$$= H(k)\underbrace{H(k)^\dagger}_{U(\Lambda_k)^\dagger} e^{iP\cdot a} \underbrace{H(k)}_{U(\Lambda_k)}|\bar{p},\sigma\rangle$$

$$= H(k)\, e^{i(\Lambda_k P)\cdot a}\, \underline{|\bar{p},\sigma\rangle}$$

$$= H(k)\, e^{i(\Lambda_k \bar{p})\cdot a}\, |\bar{p},\sigma\rangle$$

$$= e^{ik\cdot a}\, |k,\sigma\rangle$$

Lorentz

$$\bar p \longrightarrow p \longrightarrow \Lambda p$$

$$\Lambda p =$$

$$\Lambda_p$$

$$U(\Lambda)|p,\sigma\rangle = U(\Lambda)\,H(p)\,|\bar p,\sigma\rangle$$

$$= H(\Lambda p)\,\underbrace{H^{\dagger}(\Lambda p)\,U(\Lambda)\,H(p)}_{U(\Lambda_{\Lambda p})^{-1}\,U(\Lambda)\,U(\Lambda_p)}\,|\bar p,\sigma\rangle$$

$$\bar p \longleftarrow \Lambda p \longleftarrow p \xleftarrow{\Lambda_p} \bar p$$

$$= H(\Lambda p)\,U(W)\,|\bar p,\sigma\rangle$$

$$\Lambda_{\Lambda p}^{-1} \Lambda \Lambda_p = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & W_{ij} \end{array}\right)$$

$\uparrow$ Rotation

Wigner Rotation

$$W = W(\Lambda, p)$$

$$U(\Lambda)|p,\sigma\rangle = H(\Lambda p)\,\underbrace{U(W)\,|\bar{p},\sigma\rangle}$$

$$= \underbrace{H(\Lambda p)\sum_{\sigma'}|\bar{p},\sigma'\rangle}\, D_{\sigma'\sigma}(W)$$

represents W
in spin s
irrep

$$= \sum_{\sigma'}|p,\sigma'\rangle\, D_{\sigma'\sigma}(W)$$

- $W \equiv$ Wigner rotation

- W depends on choice for Λp

- $U(\Lambda)\,|p,\sigma\rangle = H(\Lambda p)\,U(W)\,|\bar{p},\sigma\rangle$

Two main options for Λ_p

lead to two different basis choice

- spin basis
- helicity basis

- Spin basis

$$\Lambda^{spin}_{p} \equiv e^{i\vec{\eta}(p)\cdot\vec{\mathbb{K}}} \qquad \begin{cases} \vec{\eta} = \left(tgh^{-1}\left|\frac{\vec{p}}{p_0}\right|\right)\vec{n} \\ \frac{\vec{p}}{|\vec{p}|} \equiv \vec{n} \end{cases}$$

$$\left(\Lambda^{spin}_{p}\right)^{\mu}{}_{\nu}\,\bar{p}^{\nu} = p^{\mu}$$

- Pure rotations on $|p,\sigma\rangle$

$$R(\underline{\theta}) = e^{-i\vec{\theta}\cdot\hat{\vec{J}}} \qquad U(R_\theta) = e^{-i\vec{\theta}\cdot\vec{J}}$$

$$U(R_\theta)|p,\sigma\rangle = U(R)\underbrace{e^{i\vec{\eta}(p)\cdot\vec{K}}}_{H(p)}|\bar{p},\sigma\rangle$$

$$= U(R)\,e^{i\eta\cdot K}\,U(R)^{\dagger}U(R)|\bar{p},\sigma\rangle$$

$$= e^{i(R\vec{\eta})\cdot\vec{K}}\,U(R)|\bar{p},\sigma\rangle$$

$$= e^{i(R\vec{\eta})\cdot K} \sum_{\sigma'} |P, \sigma'\rangle D_{\sigma'\sigma}(R)$$

$$U(R)|p,\sigma\rangle = \sum_{\sigma'} |Rp, \sigma'\rangle D_{\sigma'\sigma}(R)$$

orbital

spin

$$J = L + S$$

like in QM

- pure boost $U(\Lambda) = e^{i\eta_\Lambda \cdot K}$

$$e^{i\vec{\eta}_\Lambda \cdot \vec{K}} |p,\sigma\rangle = e^{i\vec{\eta}_\Lambda \cdot \vec{K}} e^{i\vec{\eta}(p)\cdot K} |\bar{p},\sigma\rangle$$

$$= e^{i\eta(\Lambda p)\cdot K} \underbrace{e^{-i\eta(\Lambda p)\cdot K} e^{i\eta_\Lambda \cdot K} e^{i\eta(p)\cdot K}}_{W(\Lambda,p)} |\bar{p},\sigma\rangle$$

- $\eta_\Lambda \ll 1 \Longrightarrow$

$$W = \mathbb{1} + \frac{i(\vec{\eta}_\Lambda \wedge \vec{p})\cdot \vec{J}}{p^0 + M} + O(\eta_\Lambda^2)$$