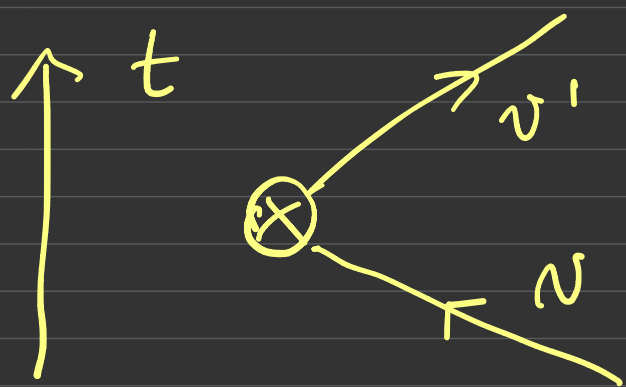
Synopsis of part I



$$\beta^\mu = (1, \vec{v})$$

$$\beta'^\mu = (1, \vec{v}')$$

$$J_\mu(x,t) = e\left[\theta(-t)\,\beta_\mu\,\delta^3(x - vt) + \theta(t)\,\beta'_\mu\,\delta^3(x - v't)\right]$$

$$J_\mu(k,t) = e\left[\theta(-t)\,\beta_\mu\, e^{-i\underline{v}\cdot\underline{k}t} + \theta(t)\,\beta'_\mu\, e^{-i\underline{v}'\cdot\underline{k}t}\right]$$

$$\Box A_\mu = J_\mu \implies i\dot{Q}_\mu - \omega_k Q_\mu = J_\mu(k,t)$$

$$\omega = k$$

$$Q_\mu(k,t)\,e^{i\omega t} = Q_\mu(k,t_0)\,e^{i\omega t_0} + ie\int_{t_0}^{t} e^{i\omega t'} J_\mu(k,t')\,dt'$$

$$e^{i\omega t}\left[Q_\mu(k,t) - \frac{\beta'_\mu}{\beta'\cdot k}\, e^{-ikv't}\right] = e^{i\omega t_0}\left[Q_\mu(k,t_0) - \frac{\beta_\mu}{\beta\cdot k}\, e^{-ikvt_0}\right]$$

$$+\left(\frac{\beta_\mu}{\beta\cdot k} - \frac{\beta'_\mu}{\beta'\cdot k}\right)$$

$$e^{i\omega t}\tilde{a}_\mu(k,t) = e^{i\omega t_0}\tilde{a}_\mu(k,t_0) + f_\mu$$

radiation field

$$\tilde{a}^{out}_\mu(k,t) = \tilde{a}^{in}_\mu(k,t_0) + f_\mu$$

$$\tilde{a}^{out}_\mu = S^\dagger a^{in}_\mu S$$

$$S = e^{\int dx_k \left(a^{\dagger in}_\mu(k) f^\mu(k) - a^{in}_\mu(k) f^{\mu *}(k)\right)}$$

$$= e^{\int dx\, a^\dagger_\mu f^\mu} e^{-\int dx\, a_\mu f^{\mu *}} e^{\frac{1}{2}\int dx\, f^*_\mu f^\mu}$$

$${}_{in}\langle 0|S|0\rangle_{in} = e^{\frac{1}{2}\int dx\, f^*_\mu f^\mu}$$

$$-\int d\Omega_k \, \epsilon^*_\mu \epsilon^\mu = A(\Delta_{uu'}) \times \int \frac{dk}{k}$$

$$\boxed{\begin{aligned} &\text{with } A(\Delta_{uu'}) = \frac{e^2}{8\pi^2}\left[\frac{2}{\Delta_{uu'}} \ln \frac{1+\Delta_{uu'}}{1-\Delta_{uu'}} - 4\right] \\ &\Delta_{uu'} = \sqrt{1 - \frac{1}{u^\mu u'_\mu}} \\ &u^\mu = \frac{1}{\sqrt{1-v^2}}(1, v) \qquad u'^\mu = \frac{1}{\sqrt{1-v'^2}}(1, v') \end{aligned}}$$

$$\boxed{\begin{aligned} &u \neq u' \\ &A > 0 \\ \hline &u = u' \\ &A = 0 \end{aligned}}$$

but $\int \frac{dk}{k}$ divergent $\xrightarrow{\text{regulate}} \int_\epsilon^\Lambda \frac{dk}{k} = \ln \frac{\Lambda}{\epsilon}$

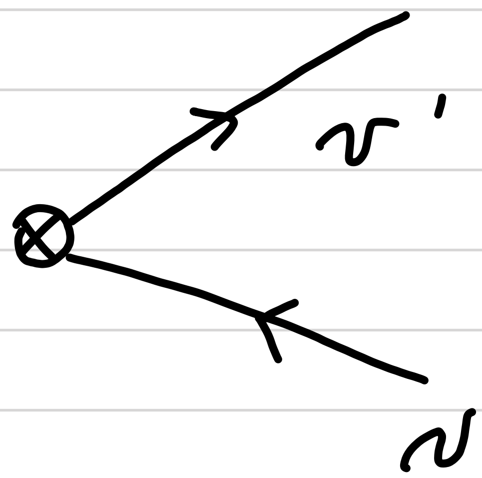
- Λ = physical $\Rightarrow$ no worry
- ϵ = we would naively like to $\longrightarrow 0$

Then

$$_{in}\langle 0 | S | 0 \rangle_{in} = e^{-\frac{A}{2} \ln \frac{\Lambda}{\epsilon}} = \left(\frac{\epsilon}{\Lambda}\right)^{A/2}$$

probability of not emitting any photon $\longrightarrow 0$ as $\epsilon \longrightarrow 0$

- Photon emission by scattered classical particle



$$J^0 = e\, \delta^3(x - \theta(t) v' t - \theta(-t) v t)$$

$$\vec{J} = e\left(\theta(t) \vec{v}'\, \delta^3(x - v' t) + \theta(-t) \vec{v}\, \delta^3(x - v t)\right)$$

- Heisenberg picture field in Lorentz gauge satisfies $\boxed{\Box A_\mu = J_\mu}$

$$\Pi_\mu \equiv -\dot{A}_\mu \implies -\dot{\Pi}_\mu - \nabla^2 A_\mu = J_\mu$$

in Fourier space

$$A_\mu(k) = \frac{1}{2\omega}\left(a_\mu(k) + a_\mu(-k)^\dagger\right)$$

$$\Pi_\mu(k) = \frac{i}{2}\left(a_\mu(k) - a_\mu(-k)^\dagger\right)$$

$$\omega_k = k \equiv |\vec{k}|^2$$

$$\Downarrow$$

$$(*) \qquad \boxed{i\dot{a}_\mu(k) = \omega_k a_\mu(k) - J_\mu(k,t)}$$

Solution

$$a_\mu(k,t) = e^{-i\omega(t-t_0)} a_\mu(k,t_0) + i e^{-i\omega t} \int_{t_0}^{t} e^{i\omega t'} J_\mu(k,t') dt'$$

$$\lim_{t_0 \to -\infty} a_\mu(k,t_0) \equiv e^{-i\omega t_0} a_\mu(k)_{in}$$

$$\lim_{t \to \infty} a_\mu(k,t) \equiv e^{-i\omega t} a_\mu(k)_{out}$$

so that

$$a_\mu(k)_{out} = a_\mu(k)_{in} + i \int_{-\infty}^{\infty} e^{i\omega t'} J_\mu(k,t') dt'$$

$$= a_\mu(k)_{in} + f_\mu(k) \qquad (**)$$

$$\Rightarrow f_\mu(k) = \int e^{i\omega_k t' - \vec{k}\vec{x}} J_\mu(x,t')$$

We find

$$f_0 = -e\left(\frac{1}{\omega - k\cdot v'} - \frac{1}{\omega - k\cdot v}\right)$$

$$\vec{f} = -e\left(\frac{\vec{v}\,'}{\omega - k\cdot v'} - \frac{\vec{v}}{\omega - k\cdot v}\right)$$

Photon number in final state

$$\frac{dn(k)}{d\Omega_k} = \langle 0| - a^{\mu\,+}_{out}(k)\, a_{\mu\,out}(k)|0\rangle$$

$$= -f^\mu f^*_\mu(k) = -e^2\left|\left(\frac{\beta_\mu}{\beta\cdot q} - \frac{\beta'_\mu}{\beta' q}\right)\right|^2$$

<u>def</u> $\vec{n} = \frac{\vec{k}}{|k|} = \frac{\vec{k}}{\omega}$

$$\frac{dn(k)}{d\Omega_k} = \frac{e^2}{\omega^2}\frac{\left|\vec{n}\times\left((n-v)\wedge(v'-v)\right)\right|^2}{(1-n\cdot v')^2(1-n\cdot v)^2}$$

$$\xRightarrow{v,v'\ll 1} \frac{e^2}{\omega^2}\cdot\left|v'_\perp - v_\perp\right|^2 \rightarrow \text{Larmor Formula}$$

But the crucial result is

$$dn(k) = \frac{d^3k}{2\omega^3(2\pi)^3}\, e^2\, g(\vec{n}, v, v')$$

$\hookrightarrow$ regular at $v, v' < 1$

$\Rightarrow \quad dn(n) = \frac{d\omega}{\omega} \int d^2\vec{n} \, \underbrace{(\ldots)}_{\text{regular}}$

$\Rightarrow \int dn(n)$ diverges logarithmically at $\omega \to 0$

$\boxed{\text{I}}$ An infinite number of soft photons is emitted

$\boxed{\text{II}}$ The emitted energy $\int \omega \, dn(\omega)$ is instead finite

☐ We can also phrase the result in terms of emission probabilities

$$a_{out} = a_{in} + f$$

implies the final state is a coherent state $\quad a_{out} = S^+ a_{in} S$

$$S = e^{\int d\mu \left(a_\mu^+ f^\mu - a_\mu f^{+\mu}\right)}$$

$$= e^{\frac{1}{2}\int f_\mu f^{\mu *} d\Omega_k} e^{\int d\Omega\, a_\mu^\dagger f^\mu}$$

$$\langle 0|S|0\rangle = e^{\frac{1}{2}\int f_\mu f^{*\mu} d\Omega} = e^{-\infty} = 0$$

$\Rightarrow$ the probability of purely elastic scattering is <u>zero</u> !!

- However it is practically impossible to <u>measure</u> a purely elastic scattering: we would need infinite energy resolution and be able to exclude the emission of the softest photon!

- We can instead more realistically consider the probability of emitting no photon of energy bigger than some resolution energy E $\Rightarrow$

$$P(E) = e^{-\int_E |f|^2 d\Omega_k} \propto e^{-A \ln \frac{E_{UV}}{E}} \sim \left(\frac{E}{E_{UV}}\right)^A$$

$0 < A$ comes from angular integration

⇒ Moral as long as we do not try and count soft photons we obtain a sensible result.

⚠ We would now like to study the same problem in QFT. In our previous example $\int |f^2| d\Omega$ was also UV divergent due to the fact that we treated the source classically. In the QFT treatment a UV cut off would surely be provided by the initial energy of the scattering particles. We shall proceed diagrammatically following very closely the treatment offered by Weinberg in Chapter 13, Volume 1.

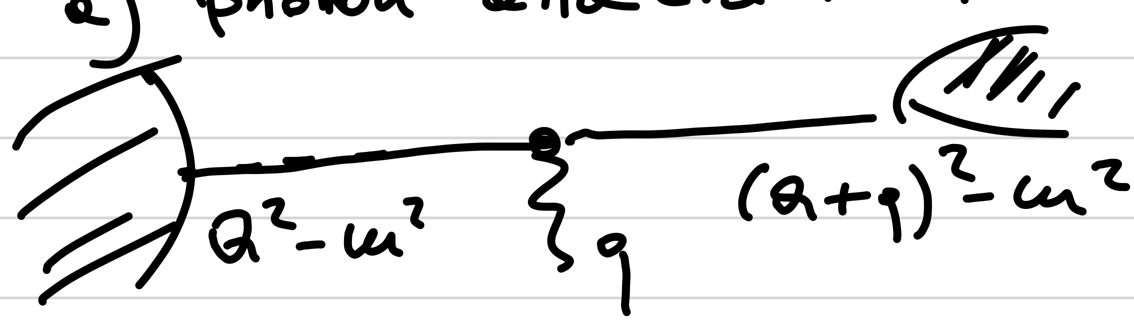
- The basic idea is to view processes as consisting of two parts:
  1) the hard scattering
  2) the soft photon emission

1) Hard scattering: diagrams where the internal lines are all largely off-shell
$\equiv$ no small denominator from internal propagators

2) Soft photons emission: starting from a hard scattering diagram I can now start attaching soft photons.

Two possibilities:

a) photon attached to internal lines

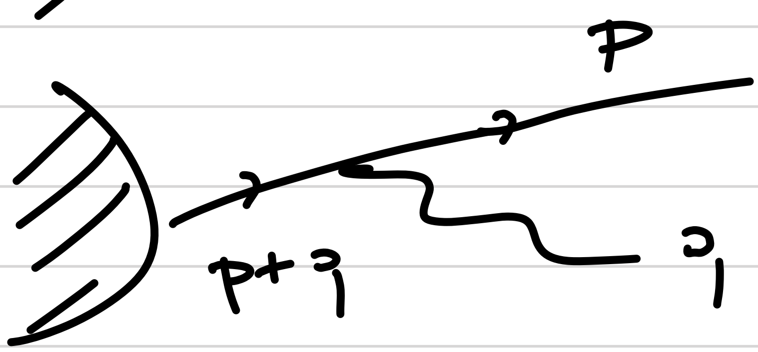


$Q^2 - m^2 \neq 0 \Rightarrow q \to 0 \Rightarrow$ no small denominator

$$|D\varphi|^2 = \partial_\mu \varphi^\dagger (-ie A^\mu)\varphi + ie A_\mu \varphi^\dagger \partial_\mu \varphi$$

$$i \cdot i \epsilon i$$

b) photon on external line



$p^2 = m^2$

<u>ex</u> scalar case

$$\frac{i}{(p+q)^2 - m^2 + i\varepsilon} \, ie\,(2p^\mu + q^\mu)$$

$$\xrightarrow{q \to 0} \quad \frac{-e\,p^\mu}{p\cdot q + i\varepsilon} = \frac{e\,p^\mu}{-pq - i\varepsilon}$$

<u>ex</u> Dirac spinor

$$\bar{u}(p,\sigma)\, ie\gamma^\mu \frac{i(\not{p} + \not{q} + m)}{(p+q)^2 - m^2 + i\varepsilon}$$

$$\bar{u}\gamma^\mu(\not{p} + \not{q} + m) = \bar{u}\left[2p^\mu + (-\not{p} + m)\gamma^\mu + \gamma^\mu \not{q}\right]$$

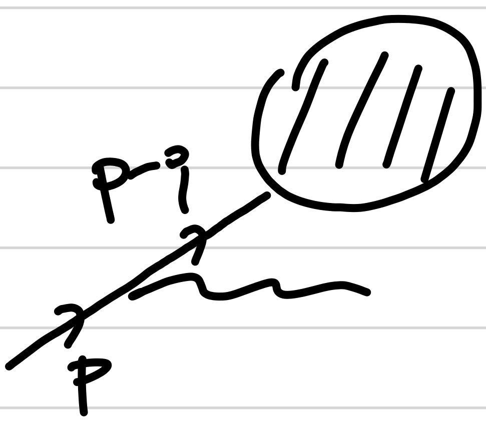
$$\overset{q \to 0}{=} \bar{u}\, 2p^\mu$$

$$= \bar{u}(p,\sigma)\, \underbrace{\frac{(-e\, p^{\mu})}{p\cdot q + i\varepsilon}}_{\text{the same factor}}$$

- One indeed finds the same factor for arbitrary spin. This indicates that soft emission reduces, for all spins, to the classical example we discussed before where the current is $\propto p^{\mu}$
- This is understandable: spin can contribute only to magnetic moment type interactions that represents coupling to $F_{\mu\nu}$ rather than to A_{μ} $\Longrightarrow$ they are suppressed in the soft limit.

○ When attaching a photon to an outgoing external line we thus find a universal result $\frac{-e p^\mu}{pq + i\varepsilon}$ which is singular at $p \to 0$

Similarly for incoming lines we find

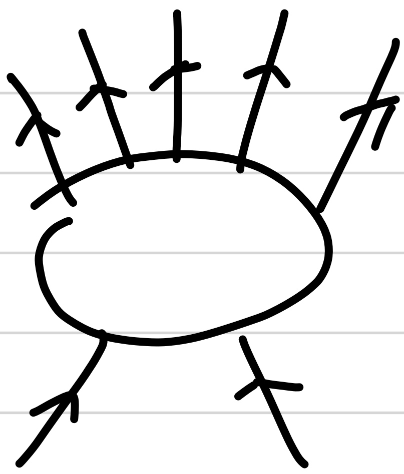


$$\frac{-e p^\mu}{-pq + i\varepsilon}$$

$$\Rightarrow \quad \frac{\eta e p}{pq - i\eta\varepsilon} \quad \begin{cases} \text{incoming}: \eta = +1 \\ \text{outgoing}: \eta = -1 \end{cases}$$

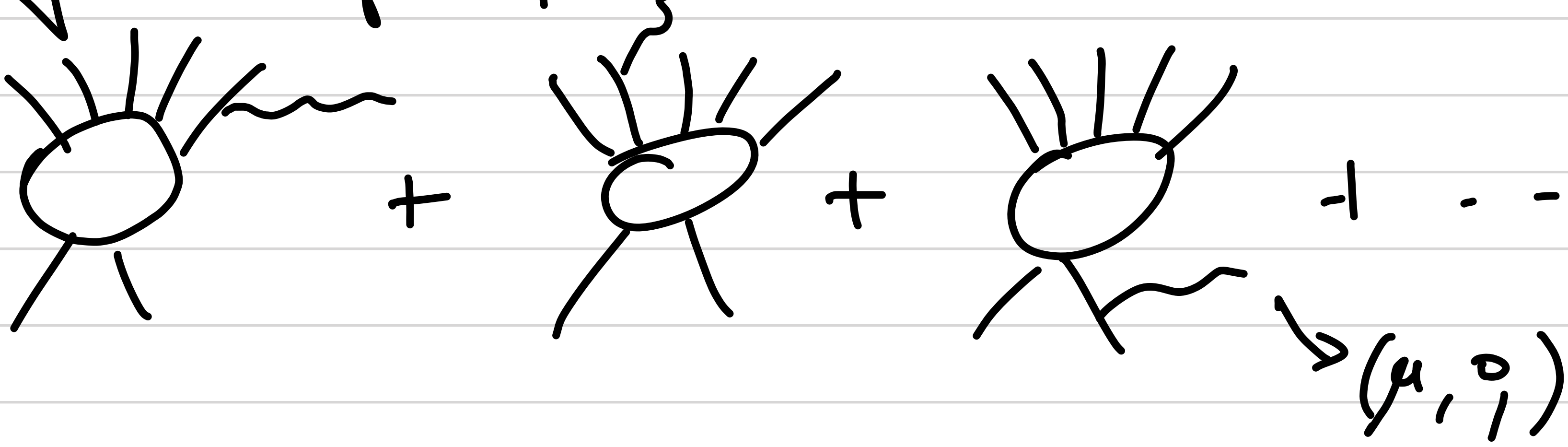
■ The emission of soft photons is dominated by external lines.

■ When considering a hard process $\alpha \to \beta$ described by



$= \mathcal{M}_{\alpha\beta}$

It is then straightforward to compute the diagrams associated to the emission of a soft photon



$$\mathcal{M}_{\alpha\beta} \to \mathcal{M}^{\mu}_{\alpha\beta} = \mathcal{M}_{\alpha\beta} \sum_n \frac{\eta_n e_n p_n^{\mu}}{p_n \cdot q - i \eta_n \varepsilon}$$

- This is a remarkable result which we obtained with very little input. In fact, in the soft limit, it is hard to think of anything else than p^μ to describe the vertex. But now we can ask

the emitted photon to be <u>indeed</u> a massless particle carrying helicities ± 1.
In that case we know that ε does not transform covariantly under Lorentz

$$\varepsilon^\mu \to \Lambda^\mu{}_\nu \varepsilon^\nu + q^\mu$$

Lorentz inv $\Longrightarrow$ $\mathcal{M}_{\alpha\beta}{}^\mu q_\mu = 0$

$$\Longleftarrow$$

$$0 = \mathcal{M}_{\alpha\beta} \sum_n \frac{\eta_n e_n \, p_n \cdot q}{p_n \cdot q - i\eta_n \varepsilon} \propto \sum_n \eta_n e_n$$

$$= \sum_{\text{incoming}} e_n - \sum_{\text{outgoing}} e_n = 0$$

Soft photon emission offers a strikingly simple proof of the relation between Lorentz invariance and (electric) charge conservation!!

- This result is so simple that it begs for a generalization to the case of particles of higher spin

Consider the case of massless spin 2

The amplitude for emitting one such soft particle will have the form

$$\mathcal{M}_{\alpha\beta}^{\mu\nu} = \mathcal{M}_{\alpha\beta} \sum_n \frac{\eta_n f_n P_n^\mu P_n^\nu}{P_n \cdot q - i\eta_n \varepsilon}$$

Like for ε^μ, one finds that $\varepsilon^{\mu\nu}$ transforms as a Lorentz tensor modulo extra terms of the form $c^\mu q^\nu + c^\nu q^\mu \Rightarrow$ to ensure Lorentz invariance it <u>must</u> be

$$\mathcal{M}_{\alpha\beta}^{\mu\nu} q_\nu = 0 \Longrightarrow \sum_n \eta_n f_n P_n^\mu \qquad (*)$$

a conservation of a linear combination

of Energy and Momentum. If (*) where independent of $P_f^\mu - P_i^\mu = 0$ then $2 \to 2$ scattering would be forbidden. In a theory consistent with a non-trivial S-matrix $f_n = f \quad \forall n$ and (*) becomes $P_f^\mu - P_i^\mu = 0$. We have thus proven that Lorentz inv + non-trivial S $\Rightarrow$ a massless spin 2 field must couple to all particles (including itself) with universal strength f !!

- We could even try and go on to higher spin. We would find the need for conservation laws that are non-linear in the momenta $\Rightarrow$

these would always imply a trivial $2 \to 2$ scattering amplitude $\Longrightarrow$ if such particles exist they can only couple with a strength $\to 0$ at $q \to 0$ $\Longrightarrow$ they cannot mediate long range forces.

Photons from classical source: complete discussion

In the study of photon emission from a accelerated classical particle we found

$$a^{\mu}_{out}(K) = a^{\mu}_{in}(k) + \underbrace{f^{\mu}(k)}_{\text{classical}}$$

We now want to more carefully derive this equation showing that a_{in}, a_{out} purely represent the radiation field, i.e. the field minus the "Coulomb" contribution. Consider a particle at rest ($J^{\mu} = e\,\delta^3(x)$). The Coulomb field is

$$\underbrace{A^{\mu}(x,t)}_{\text{coulomb}} = \frac{e}{4\pi|x|}(1,0) \Rightarrow A^{\mu}(k,\omega) = (1,0)\,\frac{2\pi e\,\delta(\omega)}{k^2}$$

. The solution for a moving particle is quickly found by performin a boost

$$A^{\mu} = \gamma(1,\vec{v})\,\frac{2\pi e\,\delta(\gamma(\omega - v\cdot k))}{k_{\perp}^2 + \gamma^2(k_{\parallel} - \omega v)^2}$$

$$k_{\perp}^2 + \gamma^2(1-v^2)^2 k_{\parallel}^2 = k_{\perp}^2 + (1-v^2)k_{\parallel}^2 = k^2 - v^2 k_{\parallel}^2$$

$$= 2\pi e \, \beta^\mu \frac{\delta(\omega - v\cdot k)}{K^2 - (v\cdot k)^2} \qquad \beta^\mu = (1, \vec{v})$$

where we used $\omega = v k_L$ in the denominator

. Fourier transforming ω

$$A^\mu(k,t) = \int A^\mu(k,\omega) e^{-i\omega t} \frac{d\omega}{2\pi} = \frac{e\beta^\mu e^{-iv\cdot kt}}{K^2 - (v\cdot k)^2}$$

$$\Pi^\mu = -\dot{A}^\mu = i v\cdot k \, A^\mu$$

$$a^\mu = K A^\mu - i\Pi^\mu = (K + v\cdot k) A^\mu = \frac{e\beta^\mu e^{-iv\cdot kt}}{K - v\cdot K}$$

$$\equiv \frac{e\beta^\mu e^{-iv\cdot kt}}{K^\nu \beta_\nu} = \frac{e u^\mu}{u^\nu K_\nu}$$

Now, in eq $*$

$$J^\mu(k,t) = e\left[\theta(t)\beta'^\mu e^{-ik\cdot v' t} + \theta(-t)\beta^\mu e^{-iv\cdot kt}\right]$$

So that the solution at finite t_0, t

takes the form

$k \equiv |\vec{k}| = \omega_k$

$$Q^\mu(\vec{k},t) = e^{-ik(t-t_0)}\left[Q^\mu(\vec{k},t_0) - \frac{e\beta^\mu}{k^\nu\beta_\nu} e^{-iv.kt_0}\right] + \frac{e\beta'^\mu}{k^\nu\beta'_\nu} e^{-iv'.kt} + e^{-ikt} f^\mu(k)$$

the contributions in green color represent the asymptotic Coulomb fields. It makes thus sense to define the asymptotic radiation fields:

$$Q^\mu_{in}(k) \underset{t_0<0}{=} e^{ikt_0}\left[Q^\mu(k,t_0) - \frac{e\beta^\mu}{k^\nu\beta_\nu} e^{-iv.kt_0}\right]$$

$$Q^\mu_{out}(k) \overset{t>0}{=} e^{ikt}\left[Q^\mu(k,t) - \frac{e\beta'^\mu}{k^\nu\beta'_\nu} e^{-iv'.kt}\right]$$

So that

$$Q^\mu_{out}(k) = Q^\mu_{in}(k) + f^\mu(k)$$

as promised.

$f|0\rangle = Q_{out}|0\rangle = S^\dagger Q_{in} S|0\rangle \quad Q_{in} S|0\rangle = f S|0\rangle$

Photon emission probability : details

$$f_0 = -e\left(\frac{1}{\omega - k\cdot v'} - \frac{1}{\omega - k\cdot v}\right)$$

$$\vec{f} = -e\left(\frac{\vec{v}'}{\omega - k\cdot v'} - \frac{\vec{v}}{\omega - k\cdot v}\right)$$

this is more conveniently written by defining the covariant 4-velocity

$$u_\mu \equiv \frac{\beta_\mu}{\sqrt{\beta^\mu \beta_\mu}} = \frac{1}{(1-v^2)^{1/2}}\,\beta_\mu$$

$$f^\mu = +e\left(\frac{u^\mu}{k^\mu u_\mu} - \frac{u'^\mu}{k^\mu u'_\mu}\right)$$

Photon number in final state

$$\frac{dn(k)}{d\Omega_k} = -\langle 0|\, a^{\mu\,+}_{out}(k)\, a_{\mu\, out}(k)\,|0\rangle$$

$$= -f^\mu f^*_\mu(k)$$

The crucial integral has the form

$$\int \frac{u\cdot u'}{(k\cdot u)(k\cdot u')} \, d\Omega_k \equiv I(u, u')$$

- The IR div can be cancelled in many ways
  Ex photon mass or energy cut-off $\omega_k > \varepsilon$
- The leading divergent piece is Lorentz inv.
- So It can be conveniently computed by going to the frame where one of the particles is at rest. Ex

$$u' = (1, 0) \equiv \bar{u}'$$
$$u = (1, \bar{v}) \frac{1}{\sqrt{1-\bar{v}^2}} \equiv \bar{u}$$

$$\Rightarrow I(\bar{u}, \bar{u}') = \int \frac{1}{1-\bar{v}\cos\theta} \frac{d\Omega_{\theta,\varphi}}{(2\pi)^3} \frac{dk}{2k}$$

$$= \frac{1}{8\pi^2} \frac{1}{\bar{v}} \ln\frac{1+\bar{v}}{1-\bar{v}} \times \int \frac{dk}{k}$$

$\bar{v}$ so obtained is an invariant velocity difference $\Delta_{uu'}$, which in an arbitrary frame can be written as

$$|\bar{v}| \equiv \Delta_{uu'} = \sqrt{1 - \frac{1}{u \cdot u'}}$$

$$= \sqrt{1 - \frac{\sqrt{(1-v^2)(1-v'^2)}}{(1 - v \cdot v')}}$$

Indeed for $v' = 0$ $\Delta_{uu'} = |v|$

- Notice that for $\Delta_{uu'} = 0$ $\frac{1}{\Delta} \ln \frac{1+\Delta}{1-\Delta} \to 2$
- In the end we thus have

$$-\int f^*_\mu f^\mu \, d\Omega_k = \underbrace{-\frac{e^2}{8\pi^2}\left(4 - \frac{2}{\Delta_{uu'}} \ln \frac{1+\Delta_{uu'}}{1-\Delta_{uu'}}\right)}_{A(\Delta_{uu'}) \geq 0 \text{ for } 1 > \Delta \geq 0} \times \int \frac{dk}{k}$$

$A(0) = 0$ expectedly

· Consider for instance an ultrarelativistic elastic collision where $|v^2| = |v'|^2 = 1 - \frac{1}{\gamma} \sim 1$

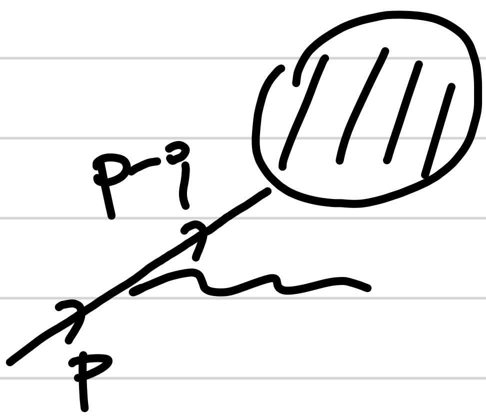
$v \cdot v' \neq 1$

$$\Delta_{uu'} = \left(1 - \frac{1}{\gamma^2} \frac{1}{1 - vv'}\right)^{1/2} = 1 - \frac{\delta}{2} \sim 1$$

$$\Rightarrow A(1 - \delta/2) = \frac{e^2}{8\pi^2}\left(2 \ln \frac{4}{\delta} - 4\right) \times \int \frac{dk}{k}$$

<u>collinear div</u> $\qquad$ <u>IR div</u>

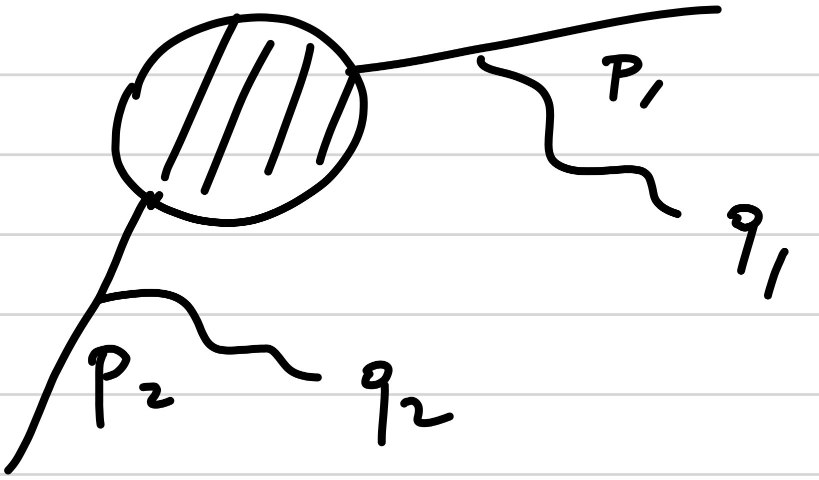
Soft radiation in QFT



$$\frac{-e\,p^{\mu}}{-pq+i\varepsilon}$$

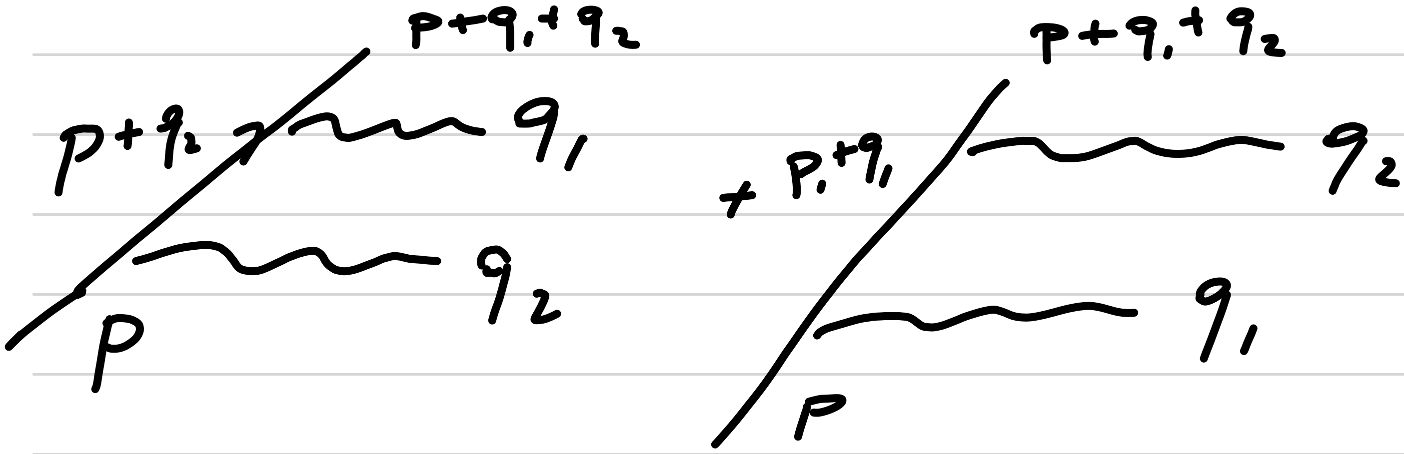
$$\Rightarrow \quad \frac{\eta\, e\, p}{pq-i\eta\varepsilon} \qquad \begin{cases} \text{incoming}: \ \eta=+1 \\ \text{outgoing}: \ \eta=-1 \end{cases}$$

- We now want to consider the case of multiple emission. It is clear that by attaching the additional photons to other lines we simply obtain the same factor:



$$\frac{\eta_1 e_1 p_1^{\mu_1}}{p_1 q_1 - i\eta_1\varepsilon}\cdot\frac{\eta_2 e_2 p_2^{\mu_2}}{p_2 q_2 - i\eta_2\varepsilon}$$

· More subtle is the case of emission from the same line. We must sum two diagrams



$$\frac{e p^\mu}{p\cdot q_2 - i\varepsilon} \cdot \frac{e p^\nu}{p\cdot(q_1+q_2) - i\varepsilon} + \frac{e p^\nu \cdot e p^\mu}{(p\cdot q_1 - i\varepsilon)(p\cdot(q_1+q_2)-i\varepsilon)}$$

$$= \frac{(e p^\mu)(e p^\nu)}{(p\cdot q_2 - i\varepsilon)(p\cdot q_1 - i\varepsilon)}$$

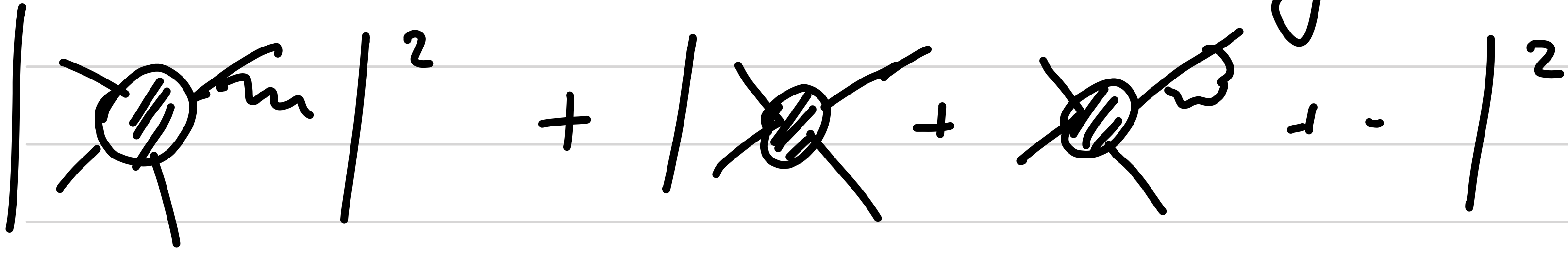
$\Rightarrow$ it works again!

Indeed, by an inductive argument, this result can be shown to generalize to the case of an arbitrary number of emissions from the same line. Given the hard matrix element

$\mathcal{M}_{\alpha\beta}$, the amplitude for N soft photon emission takes thus the form

$$\mathcal{M}_{\alpha\beta}^{\mu_1\ldots\mu_N} \longrightarrow \mathcal{M}_{\alpha\beta} \prod_{r=1}^{N} \sum_n \frac{\eta_n e_n p_n^{\mu_r}}{p_n\cdot q_r - i\eta_n \varepsilon}$$

- We can now employ this result to compute emission rates. In so doing, given the sizeable rate for emission, it will be important to include virtual effects so as to ensure <u>unitarity</u>: the soft photons will be both real and virtual. Diagrammatically



- Notice that in the case of emission from a classical source we solved the photon dyna=

mics exactly. We found the S-matrix in terms of a coeherent state operator, so that "Unitarity" was automatically preserved.

- **Virtual photons**. The diagrams involving the exchange of N virtual photons are obtained from $M_{\alpha\beta}^{\mu_1 \dots \mu_{2N}}$ by tying up each (μ_i, μ_j) pair with a photon propagator. In so doing, however we shall count each diagram $N!\,2^N$ times, corresponding to the permutation of the N propagators and of their endpoints.

$$\Rightarrow$$

$$\frac{1}{N!\,2^N} \cdot \sum_{\pi} M_{\alpha\beta}^{\mu_{\pi(1)} \dots \mu_{\pi(2N)}} \prod_{r=1}^{N} -i\,\eta_{\mu_{\pi(2r)}\mu_{\pi(2r-1)}} \underbrace{\int \frac{1}{q_{2r}^2 - i\varepsilon} \frac{d^4 q_{2r}}{(2\pi)^4}}_{G_r}$$

$$= \frac{1}{N!} \frac{1}{2^N} M_{\alpha\beta} \prod_{r=1}^{N} -i\,\eta_{\mu_{\pi(2r)}\mu_{\pi(2r-1)}} G_r \left(\sum_n \frac{\eta_n e_n p_n^{\mu_{\pi(2r-1)}}}{p_n \cdot q_r - i\eta_n \varepsilon} \right)$$

$$\cdot \sum_m \frac{\eta_m e_m p_m^{\mu_{\pi(2r)}}}{-p_m \cdot q_r - i\eta_m \varepsilon}$$

$$= \frac{\mathcal{M}_{\alpha\beta}}{N!\,2^N} \prod_{r=1}^{N} \left(\sum_{n,m} \frac{-i\eta_n \eta_m e_n e_m \, p_n \cdot p_m}{(p_n \cdot q_r - i\eta_n \varepsilon)(-p_m q_r - i\eta_m \varepsilon)} \cdot G_r \right)$$

$$= \frac{\mathcal{M}_{\alpha\beta}}{N!\,2^N} \left(\sum_{n,m} \int \frac{-i\eta_n \eta_m e_n e_m \, p_n \cdot p_m}{(p_n \cdot q - i\eta_n \varepsilon)(-p_m q - i\eta_m \varepsilon)} \frac{d^4 q}{(2\pi)^4 q^2} \right)^N$$

$$\equiv \frac{\mathcal{M}_{\alpha\beta}}{N!\,2^N} \left(\sum_{n,m} e_n e_m \eta_n \eta_m J_{n,m} \right)^N$$

$$J_{nm} = -i(p_n \cdot p_m) \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 + i\varepsilon)} \frac{1}{p_n \cdot q - i\eta_n \varepsilon} \cdot \frac{1}{-p_m \cdot q - i\eta_m \varepsilon}$$

so that, summing over N

$$\boxed{\mathcal{M}_{\alpha\beta} \longrightarrow \mathcal{M}_{\alpha\beta} \exp \frac{1}{2} \sum_{n,m} e_n e_m \eta_n \eta_m J_{nm}}$$

· Let us now compute J_{nm}.

Integrate over q_0 first. In detail

$$\frac{1}{q_0^2-\vec{q}^2+i\varepsilon}\cdot\frac{1}{p_n^0\cdot q^0-\vec{p}_n\cdot\vec{q}-i\eta_n\varepsilon}\cdot\frac{-1}{p_m^0 q^0-\vec{p}_m\cdot\vec{q}+i\eta_m\varepsilon}$$

If $\boxed{\eta_n\eta_m=-1}$ (incoming + outgoing), I can close the q_0 contour in such a way as to pick only the pole

$$q_0=|\vec{q}|-i\varepsilon \qquad (\eta_n=1 \quad \eta_m=-1)$$

$$q_0=-|\vec{q}|+i\varepsilon \qquad (\eta_n=-1 \quad \eta_m=1)$$

We thus get $\qquad (\eta_n\eta_m=-1)$

$$J_{nm}=(p_n\cdot p_m)\int\frac{d^3q}{(2\pi)^3 2q}\,\frac{1}{p_n\cdot q}\cdot\frac{1}{p_m\cdot q}$$

where $q^\mu=(q\;,\vec{q}\;)\qquad q\equiv|\vec{q}|$

Notice that this can be rewritten as

$$J_{nm} = \int \frac{d^3q}{(2\pi)^3 2q} \frac{u_n \cdot u_m}{u_n \cdot q \; u_m \cdot q} = I(u_n, u_m)$$

with $u_n^\mu \equiv p^\mu / m_n$ the 4-velocity. this is precisely the expression $I(u_n, u_m)$ already encountered in our computation with a classical source: all information about the mass of the charged particles has indeed disappeared. All that remains is the current associated to their trajectories. We can now do the computation in the case $\eta_n \eta_m = 1$.

Now we can't help picking a contribution from the other poles. After the angular integral that will give

$$J_{nm} = \int \frac{d^3q}{(2\pi)^3 2q} \frac{u_n \cdot u_m}{(u_n \cdot q)(u_m \cdot q)} - \frac{i}{4\pi} \frac{1}{\beta_{nm}} \int \frac{dq}{q}$$

$$\beta_{nm} = \sqrt{1 - \frac{1}{u_n \cdot u_m}}$$

the additional purely imaginary term will simply provide an (IR divergent) pure phase to the scattering amplitude. This has no consequence on the computation of the rate. Neglecting this phase the only relevant object $I(u_n, u_m)$ was already computed

$$I(u_n, u_m) = \frac{1}{8\pi^2} \frac{1}{\beta_{nm}} \ln \frac{1+\beta_{nm}}{1-\beta_{nm}} \times \ln \frac{\Lambda}{\lambda}$$

$\Lambda \equiv$ UV scale defining applicability of soft photon approximation. A convenient choice could be $\lambda \sim$ virtuality of hard scattering. However precise dependence on Λ must cancel in complete result

$\Rightarrow$ our original $M_{\alpha\beta}$ depends on Λ

as it only contains the effects of virtual photons with energy $> \Lambda$.

λ = IR cut-off. Without its introduction the S-matrix is not well defined, but the genuine observables should not depend on it.

Defining

$$A(\alpha \to \beta) = -\frac{1}{8\pi^2} \sum e_n e_m \eta_n \eta_m \frac{1}{\beta_{nm}} \ln \frac{1+\beta_{nm}}{1-\beta_{nm}}$$

We thus have that the scattering amplitude dressed by soft photons is

$$\mathcal{M}^{\lambda}_{\alpha\beta} = \mathcal{M}^{\Lambda}_{\alpha\beta} \left(\frac{\lambda}{\Lambda}\right)^{\frac{A(\alpha\to\beta)}{2}}$$

So that the rate is

$$\Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\Lambda}_{\alpha\beta} \left(\frac{\lambda}{\Lambda}\right)^{A(\alpha\to\beta)}$$

clearly the Λ dependence must cancel between Γ^Λ and $(\lambda/\Lambda)^A$.

- Notice that $A \geq 0$, so that soft photon emission reduces, as expected, the rate for elastic scattering.

- For the special case where only one charged particle is present in the initial and final state we find precisely the same result we found for the emission from 1 classical particle.

$$A = -\frac{e^2}{8\pi^2}\left[4 - \frac{2}{\beta}\ln\frac{1+\beta}{1-\beta}\right]$$

- This shows that, at leading order, in soft photon emission we treat the charged particles as purely classical sources.

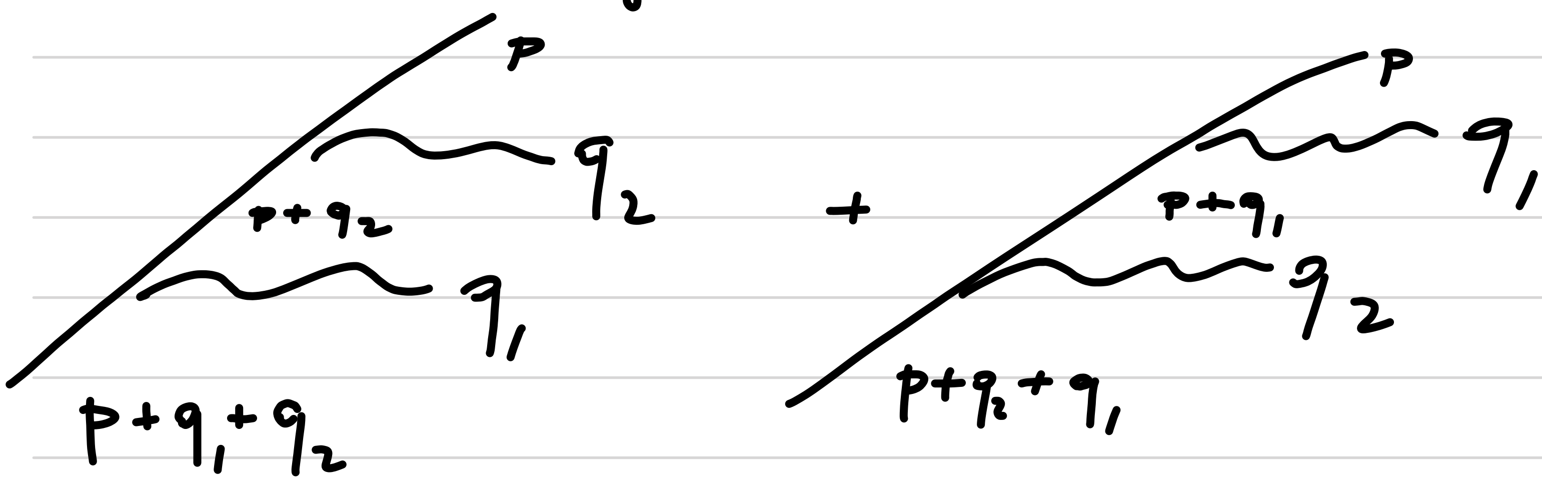
The quantum nature of the emitters is eliminated by neglecting higher powers of q (in propagators and vertices). These higher powers thus encode the genuine QFT nature of the emitters.

A puzzle on external legs.

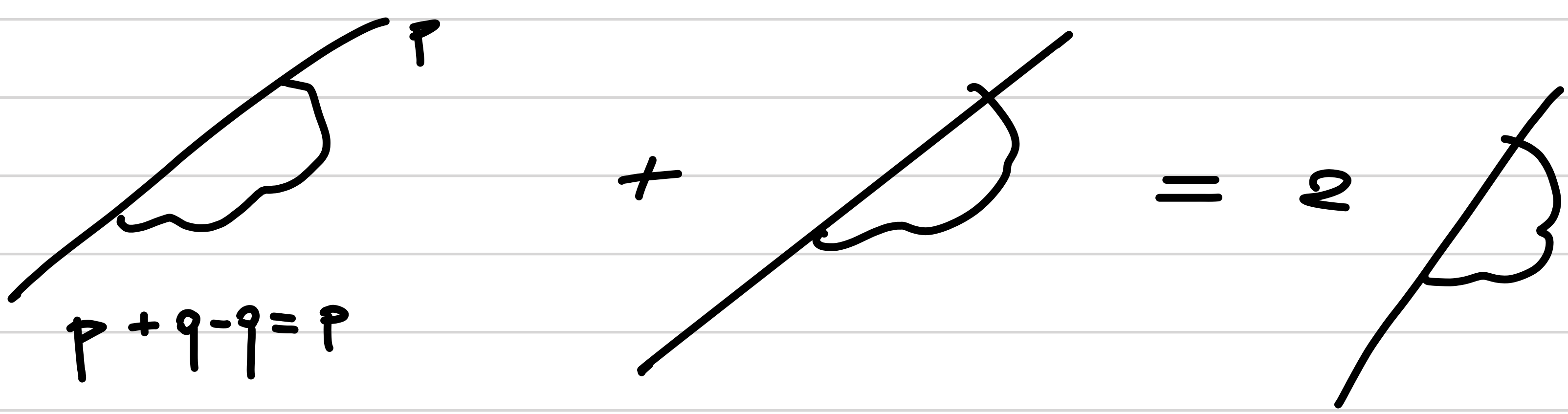
Notice that in summing over virtual photons we also, unavoidably, added diagrams that correspond to corrections to external legs. Now, one procedure of renormalization is to choose the wave function terms $(Z_n - 1)$ in such a way that the renormalized field propagator has residue $= 1$ at the particle pole $(p_n^2 = m_n^2)$. In that case, the corrections on external legs are precisely cancelled by the counterterms, so that we can disregard them. However in the computation as we organized it, we have not yet integrated over IR quantum fluctuations. In these circumstance the fields ψ_r do not have a propagator with residue $=1$ at the pole. We must thus account for corrections on external legs. Now, consistently with our soft photon approximation, we should only consider those effects that are IR divergent. As our explicit study of QED shows these are the wave function renormalizations. Mass renormalization is IR

finite (formally dominated by quantum fluctuation of scale $\sim m \gtrsim \lambda$). As the LSZ formula shows however, the wave function renormalization on the external leg should not be taken "in toto", i.e. Z_r, but only "in medio", i.e. $\sqrt{Z_r}$.
For istance, at 1-loop that corresponds to adding a $\frac{1}{2}$ factor to external leg diagrams. The question then is:
Does our computation take care of these non-trivial factors on external legs? Apparently not !!
Consider indeed what we did for the case of one virtual photon focussing on one particular external line

· At the level of emission we had two diagrams

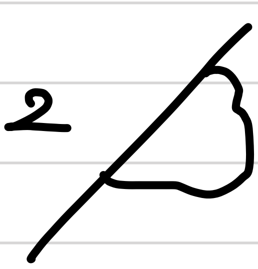
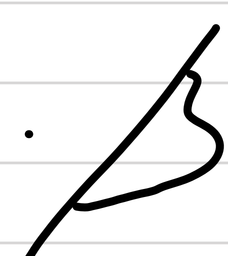


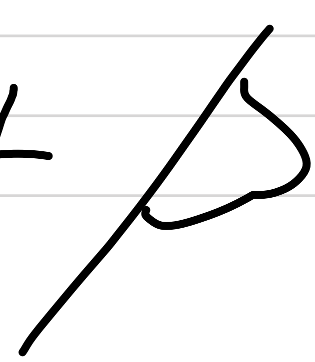
we then contracted the two photon lines



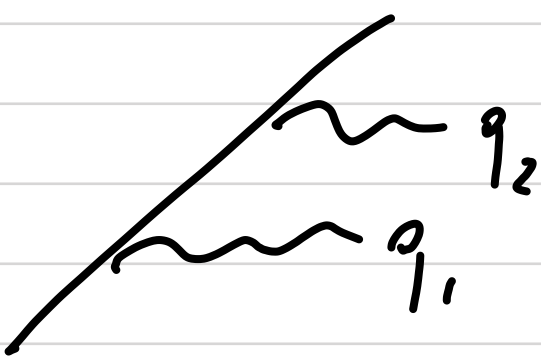
and we then divided by $\frac{1}{N!\,2^N} = \frac{1}{2}$ $(N=1)$

In the end it <u>seems</u> we took in to account of the external self-energy:

$\frac{1}{2} \times 2$  $= 1 \cdot$ 

rather than $\frac{1}{2}$ 

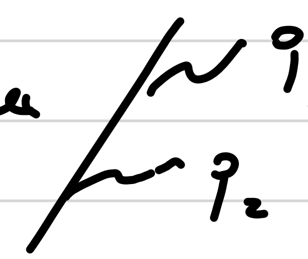
• However, and luckily, the self-energy diagrams, as we wrote them, are not well defined: in the contracted case $q_1 = -q_2$



$$= \frac{1}{p(q_1+q_2)} \cdot \frac{1}{p\, q_2}$$

$$= \frac{1}{0} \cdot \frac{1}{p \cdot q_2}$$

• By summing over the crossed diagram



we got a finite result, but we cannot be completely sure of what we did. In fact our procedure of adding ill defined quantities magically gives the needed $1/2$!

This can be seen by taking the external

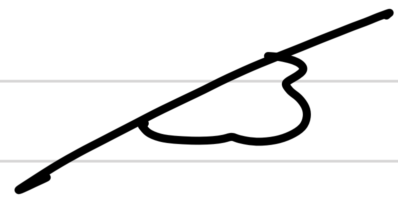
momentum p of shell, letting $q_1+q_2 \to 0$ and taking the $p^2 \to m^2$ at the end.
Defining $p^2-m^2=\Delta$ the two emission diagrams become

$$\frac{2}{2p(q_1+q_2)+\Delta}\cdot\frac{2}{2pq_2+\Delta}+\frac{2}{2p(q_1+q_2)+\Delta}\,\frac{2}{2pq_1+\Delta}$$

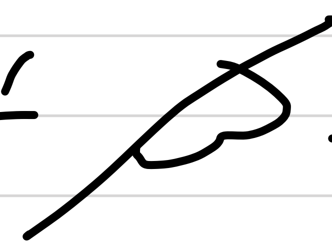
$$=\frac{4}{2p(q_1+q_2)+\Delta}\left[\frac{2p(q_1+q_2)+2\Delta}{(2pq_2+\Delta)(2pq_1+\Delta)}\right]$$

$$\xrightarrow{q_1+q_2=0}\quad \frac{2}{(pq+i\varepsilon)(-pq+i\varepsilon)} = 2\cancel{\Delta}$$

We thus find that the factor $\frac{1}{(pq)(-pq)}$ which comes from summing two ill-defined diagrams does indeed correspond to just one self energy diagram:

$$\frac{1}{(pq)(-pq)} =$$  $$= \delta Z$$

So that the adding the $1/2$ we have

 $+ \frac{1}{2}$  $$= 1 + \frac{1}{2}\delta Z \simeq Z^{1/2}$$

as it should.

It is apparently non-trivial to prove that this result, namely the proper factors, appear at higher orders.

But it works!

Real soft photon emission

We want to here compute the rate of emission of N photons (N arbitrary) such that, given E_i their energies, we have

$$E_i \leq \varepsilon \equiv \text{small} \ , \quad \sum E_i \leq E_T \qquad (\varepsilon \leq E_T)$$

- Consider the emission of N photons . In practice we should take the amplitude for arbitrary $2N_v + N$ photons, take $2N_v$ of them to be virtual and sum over N_v. This sum will give us the exponential factor we previously derived, with its Λ dependence. So that the amplitude for N soft photon emission is

$$\mathcal{M}^{\lambda}_{\alpha\beta}(q_1,\varepsilon_1; \dots q_N,\varepsilon_N) = \mathcal{M}^{\lambda}_{\alpha\beta} \prod_{r=1}^{N} \sum_n \frac{\eta_n e_n \, p_n \cdot \varepsilon_r^*}{p_n \cdot q_r}$$

Squaring the amplitude, summing over photon polarizations, and using

$\varepsilon^*_\mu \varepsilon_\nu = -\eta_{\mu\nu}$ at the end of gauge invariance, and dividing by $N!$ we get the differential rate

$$d\Gamma^\lambda_{\alpha\beta} = \frac{\Gamma^\lambda_{\alpha\beta}}{N!} \prod_r^N \frac{d^3 q_r}{(2\pi)^3 2 q_r} \sum_{n,m} \frac{-\eta_n \eta_m e_n e_m p_n \cdot p_m}{(p_n \cdot q)(p_m \cdot q)}$$

performing the angular integral we reencounter precisely the A factor we found for virtual emission:

$$d\Gamma^\lambda_{\alpha\beta} = \frac{\Gamma^\lambda_{\alpha\beta}}{N!} A(\alpha \to \beta)^N \frac{dq_1}{q_1} \cdots \frac{dq_N}{q_N}$$

· which we now must integrate with the constraint

$q_i < \varepsilon \qquad \sum q_i < E_T$

· Notice that we must still require $q_i > \lambda$

our IR cut off, which should be thought as the smallest energy scale in physics

- In the limit $\lambda \ll E$, $\varepsilon \sim E_T$ one is easily convinced that

$$\int_{\substack{\lambda < q_i < \varepsilon \\ \Sigma q_i < E_T}} \frac{dq_1}{q_1} \cdots \frac{dq_N}{q_N} \simeq \left(\ln \varepsilon/\lambda\right)^N \left(1 + O\left(\frac{1}{\ln \varepsilon/\lambda}\right)\right)$$

Summing the series of leading logs

$$\Gamma^{\lambda}_{\alpha\beta}(\varepsilon \sim E_T) = \Gamma^{\lambda}_{\alpha\beta}\, e^{A(\alpha\to\beta)\ln \varepsilon/\lambda}$$

$$= \left(\frac{\varepsilon}{\lambda}\right)^{A(\alpha\to\beta)} \Gamma^{\lambda}_{\alpha\beta}$$

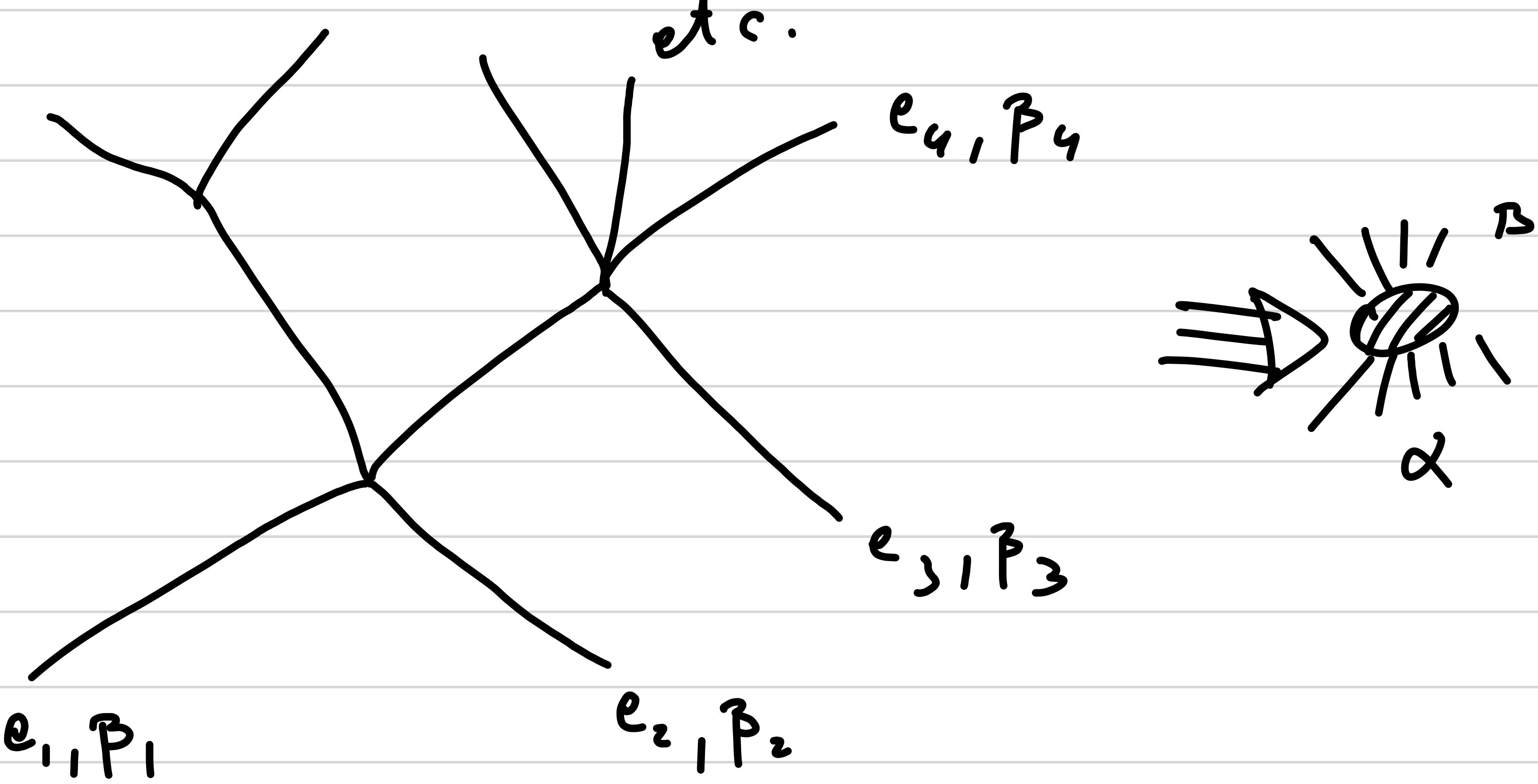
but in turn we had $\Gamma^{\lambda}_{\alpha\beta} = \left(\frac{\lambda}{\Lambda}\right)^{A(\alpha\to\beta)} \Gamma^{\Lambda}_{\alpha\beta}$

So in the end

$$\Gamma_{\alpha\beta}(\varepsilon \sim E_T) \equiv \Gamma^{\lambda}_{\alpha\beta}(\varepsilon \sim E_T) = \left(\frac{\varepsilon}{\Lambda}\right)^{A(\alpha\to\beta)} \Gamma^{\Lambda}_{\alpha\beta}$$

- this shows explicitly that IR divergences, that is the dependence on λ, cancel out when computing genuinely measurable physical quantities such as $\Gamma_{\alpha\beta}(E \sim E_T)$
- what we found is a special instance of the Kinoshita-Lee-Nauenberg (KLN) theorem, stating IR div cancel out when summing over all energy degenerate final states.
- Notice our result coincides with our computation with a classical source for the case of just one initial and one final particle. However the coincidence goes beyond as one can now easily see. We could

have considered a number of particle trajectories crossing at points in order to conserve charge locally



this diagram defines the hard process $\alpha \to \beta$ as long as the intersection points are close on the scale of the final state frequencies.