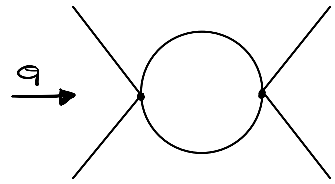
HW 8 DIMENSIONAL REGULARIZATION FILIPPO NARDI

1. Let us compute the diagram

We use the definition

$$\tilde{\lambda} = \mu^{4-d}\lambda$$

$B(q^2) =$



in dimensional regularization.

We combine the denominators using the Feynman parameters

$$B(q^2) = \frac{\tilde{\lambda}^2}{2}\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{\ell^2+m^2}\frac{1}{((\ell+q)^2+m^2)} =$$

$$= \frac{\tilde{\lambda}^2}{2}\int\frac{d^d\ell}{(2\pi)^d}\int_0^1 dx\,\frac{1}{(\ell^2+m^2+(q^2+2\ell\cdot q)x)^2} =$$

$$= \frac{\tilde{\lambda}^2}{2}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2+2\ell\cdot qx+x^2q^2-x^2q+q^2x+m^2)^2} =$$

$$= \frac{\tilde{\lambda}^2}{2}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{((\ell+qx)^2+x(1-x)q^2+m^2)^2} =$$

$$= \frac{\tilde{\lambda}^2}{2}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2+m^2+x(1-x)q^2)^2} =$$

We shift the variable $\ell+xq \to \ell$

$$= \frac{\tilde{\lambda}^2}{2}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2+\Delta)^2} = \frac{\tilde{\lambda}^2}{2}\int_0^1 dx\,\frac{\Delta^{d/2}}{\Delta^2}\int\frac{d^dy}{(2\pi)^d}\frac{1}{(y^2+1)^2}$$

$$\Delta = m^2+x(1-x)q^2$$

$$= \frac{\tilde{\lambda}^2}{2}\int_0^1 dx\,\frac{\Delta^{\frac{d}{2}-2}}{(2\pi)^d}\,\Omega_d\int_0^{+\infty}dy\,y^{d-1}(y^2+1)^{-2} =$$

$$= \frac{\tilde{\lambda}^2}{4}\frac{\Omega_d}{(2\pi)^d}\int_0^1 dx\,\Delta^{\frac{d}{2}-2}\int_0^{+\infty}dz\,z^{\frac{d}{2}-1}(z+1)^{-2} =$$

$z = y^2$

β- function

$$= \frac{\tilde{\lambda}^2}{4}\frac{\Omega_d}{(2\pi)^d}B\left(\frac{d}{2},2-\frac{d}{2}\right)\int_0^1 dx\,\Delta^{\frac{d}{2}-2}$$

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$$

Therefore, we have, in d dimensions,

$$B(q^2) = \frac{\lambda^2}{2}\frac{\mu^{2(4-d)}}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\ \Delta^{\frac{d}{2}-2}$$

We now take the limit $d \to 4-\varepsilon$.

$$B(q^2) = \frac{\lambda^2}{2}\frac{\mu^{2\varepsilon}}{16\pi^2}(4\pi)^{+\varepsilon/2}\Gamma(\varepsilon/2)\int_0^1 dx\ \Delta^{-\varepsilon/2} =$$

$$= \frac{\lambda^2}{2}\frac{\mu^{\varepsilon}}{16\pi^2}\left(\frac{2}{\varepsilon}-\gamma+O(\varepsilon)\right)\int_0^1 dx\left(1-\frac{\varepsilon}{2}\ln\left(\frac{\Delta}{4\pi\mu^2}\right)+O(\varepsilon^2)\right)$$

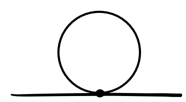
$$= \frac{\lambda^2}{2}\frac{\mu^{\varepsilon}}{16\pi^2}\left(\frac{2}{\varepsilon}-\gamma-\int_0^1 dx\ \ln\left(\frac{m^2+x(1-x)q^2}{4\pi\mu^2}\right)+O(\varepsilon)\right)$$

Mathematica

$$\downarrow$$

$$= \frac{\lambda^2\mu^{\varepsilon}}{32\pi^2}\left(\frac{2}{\varepsilon}-\gamma+\ln\left(\frac{4\pi\mu^2}{m^2}\right)+2-2\sqrt{1+\frac{4m^2}{q^2}}\ \mathrm{Arctanh}\left(\sqrt{\frac{q^2}{q^2+4m^2}}\right)\right)$$

2. Let us now compute the diagram



$$T = -\frac{\tilde{\lambda}}{2}\int\frac{d^d l}{(2\pi)^d}\frac{1}{l^2+m^2} = \frac{\tilde{\lambda}}{2}m^{d-2}\int\frac{d^d y}{(2\pi)^d}\frac{1}{y^2+1} =$$

$$= \frac{\tilde{\lambda}}{2}m^{d-2}\frac{\Omega_d}{(2\pi)^d}\int_0^{+\infty}dy\ y^{d-1}(y^2+1)^{-1} =$$

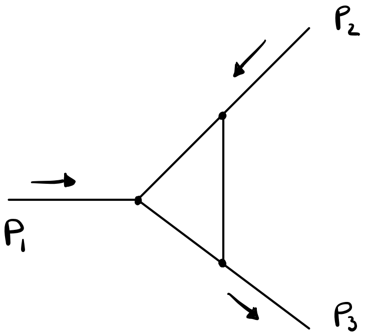
$$= \frac{m^{d-2}}{2}\frac{\lambda\mu^{4-d}}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)$$

In $d=4-\varepsilon$ we find

$$T = -\frac{m^2}{2}\frac{\lambda}{16\pi^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\varepsilon/2}\left(-\frac{2}{\varepsilon}+\gamma-1+O(\varepsilon)\right) =$$

$$= -\frac{m^2\lambda}{32\pi^2}\left(-\frac{2}{\varepsilon}+\gamma-1+\ln\left(\frac{m^2}{4\pi\mu^2}\right)+O(\varepsilon)\right)$$

3. We want to compute the divergent part of the integral corresponding to the Feynman diagram



$$= -\lambda^3 \mu^{(6-d)3/2} \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(p_1+\ell)^2+m^2} \frac{1}{(p_1+p_2+\ell)^2+m^2} \frac{1}{\ell^2+m^2}$$

We notice that the superficial divergence of the diagram in $d=6$ is 0 and thus we expect it to be logarithmically divergent.
In addition, we notice that the divergent part is independent of the external momenta because the derivatives of the diagram are all finite.
Thus, we compute

$$I_{div} = -\lambda^3 \mu^{(6-d)3/2} \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2+m^2)^3} =$$

$$= -\lambda^3 \mu^{(6-d)3/2} m^{d-6} \int \frac{d^dy}{(2\pi)^d} \frac{1}{(y^2+1)^3} =$$

$$= -\lambda^3 \mu^{(6-d)3/2} m^{d-6} \frac{\Omega_d}{(2\pi)^d} \int dy\, y^{d-1} (y^2+1)^{-3} =$$

$$= -\lambda^3 \mu^{(6-d)3/2} m^{d-6} \frac{\Omega_d}{(2\pi)^d} \frac{1}{2} \int_0^{+\infty} dz\, z^{\frac{d}{2}-1} (z+1)^{-3} =$$

$x = \frac{1}{z+1}$

$z = -1 + \frac{1}{x}$

$$= -\lambda^3 \mu^{(6-d)3/2} m^{d-6} \frac{\Omega_d}{(2\pi)^d} \frac{1}{2} \int_0^1 dx\, (1-x)^{\frac{d}{2}-1} x^{2-\frac{d}{2}} =$$

$$= -\lambda^3 \mu^{(6-d)3/2} m^{d-6} \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma(3)} \Gamma\left(3-\frac{d}{2}\right) =$$

$$= -\frac{\lambda^3}{2} \frac{\mu^{(6-d)/2}}{(4\pi)^{d/2}} \left(\frac{m}{\mu}\right)^{d-6} \Gamma\left(3-\frac{d}{2}\right)$$

In $d = 6-\varepsilon$ we find

$$I_{div} = -\frac{\lambda^3}{2} \frac{\mu^{\varepsilon/2}}{(4\pi)^3} \left(\frac{m^2}{4\pi\mu^2}\right)^{\varepsilon/2} \left(\frac{2}{\varepsilon} - \gamma + O(\varepsilon)\right) = -\frac{\lambda^3}{2} \frac{\mu^{\varepsilon/2}}{(4\pi)^3} \left(\frac{2}{\varepsilon} + O(\varepsilon^0)\right)$$

4. Let us consider

- $$\int d^d k \, \frac{K^\mu}{K^2(K+P)^2} = I^\mu(P)$$

If we make an $SO(d)$ transformation on P^μ we find

$$I^\mu(\Lambda P) = \int d^d k \, \frac{K^\mu}{K^2(K+\Lambda P)^2} = \int d^d k \, \frac{K^\mu}{K^2(\Lambda^{-1}K+P)^2} =$$

$$\underset{K=\Lambda q}{=} \int d^d q \, \frac{\Lambda^\mu{}_\nu q^\nu}{q^2(q+P)^2} = \Lambda^\mu{}_\nu I^\mu(P)$$

This means that

$$I^\mu(P) = P^\mu f(P^2)$$

and we can find the scalar function as follows

$$P_\mu I^\mu(P) = P^2 f(P) = \int d^d k \, \frac{P\cdot K}{K^2(K+P)^2} =$$

$$= \int d^d k \, \frac{1}{2}\left[\frac{(P+K)^2}{K^2(K+P)^2} - \frac{P^2+K^2}{K^2(K+P)^2}\right] =$$

$$= \frac{1}{2}\int \frac{d^d k}{K^2} - \frac{P^2}{2}\int d^d k \, \frac{1}{K^2(K+P)^2} - \frac{1}{2}\int d^d k \, \frac{1}{(K+P)^2} =$$

$$= -\frac{P^2}{2}\int d^d k \, \frac{1}{K^2(K+P)^2} = -\frac{P^2}{2} I_2(P)$$

Therefore, we find

$$I^\mu(P) = -\frac{P^\mu}{2} I_2(P)$$

- $$J^{\mu\nu}(P) = \int d^d k \, \frac{K^\mu K^\nu}{K^2(K+P)^2}$$

Using the same argument as above we find $J^{\mu\nu}(\Lambda P) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma J^{\rho\sigma}(P)$

and thus

$$J^{\mu\nu}(P) = \delta^{\mu\nu} J_1(P^2) + \frac{P^\mu P^\nu}{P^2} J_2(P^2)$$

We find J_1 and J_2 noticing that

$$J^{\mu}{}_{\mu}(P) = d\, J_1(P^2) + J_2(P^2) = \int d^dk \, (k+P)^{-2} = 0$$

$$P_\mu P_\nu J^{\mu\nu}(P) = P^2 (J_1(P^2) + J_2(P^2)) = \int d^dk \, \frac{(P\cdot k)^2}{k^2(k+P)^2} =$$

$$= \frac{1}{4} \int d^dk \, \frac{[(P+k)^2 - P^2 - k^2]^2}{k^2(k+P)^2}$$

$2(P\cdot k) = (P+k)^2 - P^2 - k^2$

$$= \frac{1}{4} \int d^dk \, \frac{[(P+k)^4 + P^4 + k^4 - 2P^2(P+k)^2 + 2P^2k^2 - 2k^2(P+k)^2]}{k^2(k+P)^2} =$$

$$\int d^dk \, \frac{(P+k)^2}{k^2} = P_\mu \int d^dk \, \frac{k^\mu}{k^2} = 0$$

$$= \frac{1}{4} \int d^dk \left[\frac{(P+k)^2}{k^2} + \frac{P^4}{k^2(k+P)^2} + \frac{k^2}{(k+P)^2} \right] =$$

$$= \frac{P^4}{4} I_2$$

Thus, we find

$$d\,J_1 + J_2 = 0 \qquad\qquad J_1 = -\frac{1}{d-1} \frac{P^2}{4} I_2$$

$$J_1 + J_2 = \frac{P^2}{4} I_2 \qquad\qquad J_2 = +\frac{d}{d-1} \frac{P^2}{4} I_2$$

and finally

$$J^{\mu\nu}(P) = \frac{d}{4(d-1)} \left(P^\mu P^\nu - \frac{P^2}{d} \delta^{\mu\nu} \right) I_2$$