KÄLLEN-LEHMANN DECOMPOSITION, CPT, CAUSALITY AND STATISTICS

PART I

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Let us consider the states $|r, \vec{P}, S_3\rangle$ where $\vec{P}$ is the 3-momentum, S_3 the angular momentum along the $\hat{z}$ direction and r is a collective index which includes the mass of the state m_r, its angular momentum S_r but also other quantum numbers. These states are normalized as

$$\langle r', \vec{P}', S_3' | r, \vec{P}, S_3\rangle = \delta(r'-r)\,\delta_{S_3 S_3'}\,(2\pi)^3\, 2P_r^0\, \delta^{(3)}(\vec{P}'-\vec{P})$$

(i) Given the definition

$$\mathbb{P}(K,S) = \sum\!\!\!\!\!\!\int_{r,\vec{P},S_3} \delta_{S,S_r}\,\delta^{(4)}(P_r-K)\,|r,\vec{P},S_3\rangle\langle r,\vec{P},S_3|$$

let us show that $\mathbb{P}(K,S)\,\mathbb{P}(K',S') = \delta^{(4)}(K-K')\,\delta_{S,S'}\,\mathbb{P}(K,S)$

$$\mathbb{P}(K,S)\,\mathbb{P}(K',S') = \sum\!\!\!\!\!\!\int_{r,\vec{P},S_3} \delta_{S,S_r}\,\delta^{(4)}(P_r-K)\,|r,\vec{P},S_3\rangle\langle r,\vec{P},S_3|$$
$$\sum\!\!\!\!\!\!\int_{r',\vec{P}',S_3'} \delta_{S',S_{r'}}\,\delta^{(4)}(P_r'-K')\,|r',\vec{P}',S_3'\rangle\langle r',\vec{P}',S_3'| =$$

$$= \sum\!\!\!\!\!\!\int_{r,\vec{P},S_3} \sum\!\!\!\!\!\!\int_{r',\vec{P}',S_3'} \delta_{S,S_r}\,\delta^{(4)}(P_r-K)\,\delta_{S',S_{r'}}\,\delta^{(4)}(P_r'-K')\,|r,\vec{P},S_3\rangle\langle r,\vec{P},S_3|r',\vec{P}',S_3'\rangle\langle r',\vec{P}',S_3'| =$$

$$= \sum\!\!\!\!\!\!\int_{r,\vec{P},S_3} \sum\!\!\!\!\!\!\int_{r',\vec{P}',S_3'} \delta_{S,S_r}\,\delta^{(4)}(P_r-K)\,\delta_{S',S_{r'}}\,\delta^{(4)}(P_r'-K')\,\delta(r'-r)\,\delta_{S_3,S_3'}\,(2\pi)^3\,2P_r^0\,\delta^{(3)}(\vec{P}'-\vec{P})$$
$$|r,\vec{P},S_3\rangle\langle r',\vec{P}',S_3'| =$$

$$= \sum\!\!\!\!\!\!\int_{r,\vec{P},S_3} \delta_{S,S_r}\,\delta^{(4)}(P_r-K) \int dr' \int \frac{d^3P'}{(2\pi)^3}\frac{1}{2P_{r'}^{0\prime}} \sum_{S_3'=-S_r'}^{S_3'=S_r'} \delta_{S',S_{r'}}\,\delta^{(4)}(P_r'-K')\,\delta(r'-r)$$
$$\delta_{S_3,S_3'}\,(2\pi)^3\,2P_r^0\,\delta^{(3)}(\vec{P}'-\vec{P})\,|r,\vec{P},S_3\rangle\langle r',\vec{P}',S_3'| =$$

$$= \sum\!\!\!\!\!\!\int_{r,\vec{P},S_3} \delta_{S,S_r}\,\delta^{(4)}(P_r-K) \int \frac{d^3P'}{(2\pi)^3}\frac{1}{2P_r^{0\prime}} \sum_{S_3'=-S_r}^{S_3'=S_r} \delta_{S',S_r}\,\delta^{(4)}(P_r'-K')\,\delta_{S_3,S_3'}$$
$$(2\pi)^3\,2P_r^0\,\delta^{(3)}(\vec{P}'-\vec{P})\,|r,\vec{P},S_3\rangle\langle r',\vec{P}',S_3'| =$$

$$= \sum\!\!\!\!\!\!\int_{r,\vec{P},S_3} \delta_{S,S'}\,\delta_{S,S_r}\,\delta^{(4)}(P_r-K)\,\delta^{(4)}(K-K')\,|r,\vec{P},S_3\rangle\langle r,\vec{P},S_3| =$$

$$= \delta_{S,S'}\,\delta^{(4)}(K-K')\sum\!\!\!\!\!\!\int_{r,\vec{P},S_3} \delta_{S,S_r}\,\delta^{(4)}(P_r-K)\,|r,\vec{P},S_3\rangle\langle r,\vec{P},S_3| =$$

$$= \delta_{S,S'}\,\delta^{(4)}(K-K')\,\mathbb{P}(K,S)$$

In addition,

$$\sum\!\!\!\!\!\!\int_{K,S} \mathbb{P}(K,S) = \sum\!\!\!\!\!\!\int_{r,\vec{P},S_3} |r,\vec{P},S_3\rangle\langle r,\vec{P},S_3| = \mathbb{1}$$

Therefore the $\mathbb{P}(K,S)$ are projectors.

(ii) Let us consider the action of a CPT transformation on the states.

$$\Theta |r, \vec{P}, S_3\rangle = \eta_r (-1)^{S_r - S_3} |\bar{r}, \vec{P}, -S_3\rangle$$

where $|\eta_r| = 1$. We want to show that $\Theta^+ P(K,S)\Theta = P(K,S)$.

$$\Theta^+ \mathbb{P}(K,S)\Theta = \sum\!\!\!\!\!\!\int_{r,\vec{P},S_3} \delta_{S,S_r} \delta^{(4)}(K-P_r)\, \Theta^+ |r,\vec{P},S_3\rangle\langle r,\vec{P},S_3| \Theta =$$

$$= \sum\!\!\!\!\!\!\int_{r,\vec{P},S_3} \delta_{S,S_r} \delta^{(4)}(K-P_3)\, |\eta_r|^2\, |\bar{r},\vec{P},-S_3\rangle\langle \bar{r},\vec{P},-S_3| =$$

Because we are summing over all possible states

$$= \sum\!\!\!\!\!\!\int_{r,\vec{P},S_3} \delta_{S,S_r} \delta^{(4)}(K-P_3)\, |r,\vec{P},S_3\rangle\langle r,\vec{P},S_3| = \mathbb{P}(K,S)$$

(iii) We now consider a local operator $\mathcal{O}_A(x)$ transforming in the irreducible representation (j_-, j_+). Under a CPT transformation

$$\Theta^+ \mathcal{O}_A(x)\Theta = \eta_{\mathcal{O}}\, \mathcal{O}^+_{\bar{A}}(-x)$$

$(\Omega, \psi\Omega)$
$= (*)\ (\Theta\Omega, \psi\Theta\Omega)$
$= (\Omega, \Theta^+\psi\Theta\Omega)^*$

with $|\eta_{\mathcal{O}}| = 1$. We want to show that
$$\langle 0|\mathcal{O}_A(0)\mathbb{P}(K,S)\mathcal{O}^+_{\bar{B}}(0)|0\rangle = \langle 0|\mathcal{O}^+_{\bar{B}}(0)\mathbb{P}(K,S)\mathcal{O}_A(0)|0\rangle$$

Because CPT is a symmetry of the theory $\Theta|0\rangle = |0\rangle$. Thus

Antiunitarity of Θ $(*)$

$$\langle 0|\mathcal{O}_A(0)\mathbb{P}(K,S)\mathcal{O}^+_{\bar{B}}(0)|0\rangle = \langle 0|\Theta^+\mathcal{O}_A(0)\mathbb{P}(K,S)\mathcal{O}^+_{\bar{B}}(0)\Theta|0\rangle^* =$$

$$= \langle 0|\Theta^+\mathcal{O}_A(0)\Theta\Theta^+\mathbb{P}(K,S)\Theta\Theta^+\mathcal{O}^+_{\bar{B}}(0)\Theta|0\rangle^* =$$

$$= |\eta_{\mathcal{O}}|^2 \langle 0|\mathcal{O}^+_{\bar{A}}(0)\mathbb{P}(K,S)\mathcal{O}_B(0)|0\rangle^* = \langle 0|\mathcal{O}^+_{\bar{B}}(0)\mathbb{P}(K,S)\mathcal{O}_A(0)|0\rangle$$

(iv) We consider the matrix element

$$\langle 0|\mathcal{O}_A(0)|r,\vec{P},S_3\rangle = Z^{1/2}_{\mathcal{O},r}\, \psi^{(S_r)}_{A,S_3}(P_r)$$

that defines the wave function $\psi^{(S_r)}_{A,S_3}(P_r)$. We recall that

$$\langle 0|\mathcal{O}_A(0)|r,\vec{P},S_3\rangle = \langle 0|\mathcal{O}_A(0)U(\Lambda_P)|r,\vec{\bar{P}},S_3\rangle =$$

$$= \langle 0|U^+(\Lambda_P)\mathcal{O}_A(0)U(\Lambda_P)|r,\vec{\bar{P}},S_3\rangle =$$

$$= D_{AA'}(\Lambda_P)\langle 0|\mathcal{O}_{A'}(0)|r,\vec{0},S_3\rangle$$

and thus
$$\psi^{(S_r)}_{A,S_3}(P_r) = D_{AA'}(\Lambda_P)\,\psi^{(S_r)}_{A',S_3}(\bar{P}_r)$$

where $\bar{P}^\mu_r = (m_r, \vec{0})$. In addition

$$R_W = \Lambda^{-1}_{\Lambda p}\Lambda\Lambda_p$$

$$\langle 0|\mathcal{O}_A(0)U(\Lambda)|r,\vec{p},s_3\rangle = D^{(s_r)}_{s_3's_3}(R_W)\langle 0|\mathcal{O}_A(0)|r,\overrightarrow{(\Lambda p)},s_3'\rangle =$$

$$= \langle 0|U^+(\Lambda)\mathcal{O}_A(0)U(\Lambda)|r,\vec{p},s_3\rangle = D_{A,A'}(\Lambda)\langle 0|\mathcal{O}_{A'}(0)|r,\vec{p},s_3\rangle$$

that is

$$\psi^{(s_r)}_{A,s_3}(\Lambda p) = D^{(s_r)}_{s_3's_3}(R_W^{-1})D_{A,A'}(\Lambda)\psi^{(s_r)}_{A',s_3'}(p)$$

Let us now consider the projectors

$$\Pi^{(s)}_{A\bar{B}}(p) = \sum_{s_3=-s}^{+s}\psi^{(s)}_{A,s_3}(p)\psi^{(s)*}_{\bar{B},s_3}(p)$$

From the previous result we get

$$\Pi^{(s)}_{A\bar{B}}(\Lambda p) = \sum_{s_3=-s}^{+s}\psi^{(s)}_{A,s_3}(\Lambda p)\psi^{(s)}_{\bar{B},s_3}(\Lambda p) =$$

$$= D_{AA'}(\Lambda)D_{\bar{B},\bar{B}'}(\Lambda)\sum_{s_3,s_3',s_3''}D^{(s)}_{s_3's_3}(R_W^{-1})D^{(s)*}_{s_3'',s_3}(R_W^{-1})\psi^{(s)}_{A',s_3'}(p)\psi^{(s)}_{\bar{B}',s_3''}(p) =$$

Rotations are unitarely represented

$$= D_{AA'}(\Lambda)D_{\bar{B},\bar{B}'}(\Lambda)\sum_{s_3,s_3',s_3''}D^{(s)}_{s_3's_3}(R_W^{-1})D^{(s)}_{s_3,s_3''}(R_W)\psi^{(s)}_{A',s_3'}(p)\psi^{(s)}_{\bar{B}',s_3''}(p) =$$

$$= D_{AA'}(\Lambda)D_{\bar{B},\bar{B}'}(\Lambda)\sum_{s_3}\psi^{(s)}_{A',s_3}(p)\psi^{(s)}_{\bar{B}',s_3}(p) =$$

$$= D_{AA'}(\Lambda)D_{\bar{B},\bar{B}'}(\Lambda)\,\Pi^{(s)}_{A'\bar{B}'}(p)$$

(v) We want to prove

$$\Pi^{(s)}_{A\bar{B}}(p) = \Pi^{(s)}_{A\bar{B}}(-p)(-1)^{2j_-+2j_+}$$

From the previous point we see that $\Pi^{(s)}_{A\bar{B}}(p)$ transforms as a $(j_-,j_+)\otimes(j_+,j_-)$ tensor. Thus, we may write

$$\Pi^{(s)}_{A\bar{B}} \sim \Pi^{(s)}_{\alpha_1,\dots,\alpha_{2j_-+2j_+};\dot{\beta}_1,\dots,\dot{\beta}_{2j_++2j_-}}$$

Now we notice that, from the previous point we get $(\bar{p}^\mu = (m,\vec{0}))$

$$D_{AA'}(R)D_{\bar{B},\bar{B}'}(R)\,\Pi^{(s)}_{A'\bar{B}'}(\bar{p}) = \Pi^{(s)}_{A\bar{B}}(R\bar{p}) = \Pi^{(s)}_{A\bar{B}}(\bar{p})$$

This means that $\Pi^{(s)}_{A\bar{B}}(\bar{p})$ is invariant under rotations and thus it has spin zero. SO(3) spin zero representations are contained only in SO(1,3) representations of the form (j,j). This means that if we identify the irreps in $\Pi^{(s)}_{A\bar{B}}$ they must be of this form. This can be done symmetrizing and antisymmetrizing the indices as usual. The antisymmetrization produces a tensor with two indices less

$$\Pi_{[\alpha_1\alpha_2]\alpha_3 \dots ; \dots} = \varepsilon_{\alpha_1\alpha_2}\, \varepsilon^{\gamma_1\gamma_2}\, \Pi_{\gamma_1\gamma_2\alpha_3 \dots ; \dots}$$

Therefore the irreps are of the form

$$\Pi^{(s)}_{\{\alpha_1,\dots,\alpha_j\};\{\dot\beta_1,\dots,\dot\beta_j\}}(P)$$

where $j = 2j_+ + 2j_- - 2n$. This tensor can always be written as

$$\Pi^{(s)}_{\{\alpha_1,\dots,\alpha_j\};\{\dot\beta_1,\dots,\dot\beta_j\}} = \Pi_{\mu_1,\dots,\mu_j}(P) \sum_{\sigma\in\Pi(j)} \sum_{\zeta\in\Pi(j)} \sigma^{\mu_1}_{\alpha_{\sigma(1)}\dot\beta_{\zeta(1)}} \dots \sigma^{\mu_j}_{\alpha_{\sigma(j)}\dot\beta_{\zeta(j)}}$$

and from the covariance of $\Pi^{(s)}_{A\bar B}(P)$ it follows

$$\Pi_{\mu_1\dots\mu_j}(\Lambda P) = \Lambda_{\mu_1}{}^{\nu_1} \dots \Lambda_{\mu_j}{}^{\nu_j}\, \Pi_{\nu_1\dots\nu_j}(P)$$

Therefore, $\Pi(P)$ can be written as combinations of $g^{\mu\nu}$ and P^μ. If K is the number of metric tensors in one of these terms, we find that under $P \to -P$ it gets a factor $(-1)^{j-2K} = (-1)^j$. Thus,

$$\begin{aligned}\Pi_{\mu_1\dots\mu_j}(-P) &= (-1)^j\, \Pi_{\mu_1,\dots,\mu_j}(P) = \\ &= (-1)^{2j_+ + 2j_- - 2n}\, \Pi_{\mu_1,\dots,\mu_j}(P) = \\ &= (-1)^{2j_+ + 2j_-}\, \Pi_{\mu_1,\dots,\mu_j}(P)\end{aligned}$$

and finally

$$\Pi^{(s)}_{A\bar B}(P) = \Pi^{(s)}_{A\bar B}(-P)\,(-1)^{2j_- + 2j_+}$$

(vi) The sets of IN and OUT states constitute two complete basis for the Hilbert space of states (Asymptotic completeness). In addition, for a Lorentz invariant theory, it exists a set of unitary operators $U(\Lambda)$ which acts on these states. The states can be identified with the same quantum numbers used before $|r, \vec P, s_3; \text{in/out}\rangle$ and they satisfy the same normalization

$$\langle \text{in/out}; r', \vec P', s_3' | r, \vec P, s_3; \text{in/out}\rangle = \delta(r'-r)\, \delta_{s_3 s_3'}\, (2\pi)^3\, 2P_r^0\, \delta^{(3)}(\vec P' - \vec P)$$

Thus, all the properties we used to define $\mathbb{P}(K,s)$ as projectors are still valid in the case we use IN or OUT states. In particular

- $\mathbb{P}_{in/out}(K',s')\,\mathbb{P}_{in/out}(K,s) = \delta^{(4)}(K-K')\,\delta_{s,s'}\,\mathbb{P}_{in/out}(K,s)$

as a consequence of the normalization of the states and

- $\sum\!\!\!\!\!\!\int_{K,s} \mathbb{P}(K,s) = \mathbb{1}$

as a consequence of asymptotic completeness.