HW4
Wave equations

In class you showed that wave equations for massive particles manifest themselves through projectors and the onshell condition. In this exercise you are asked to find wave equations for different massive particle.

Spin $\frac{1}{2}$

In the previous homework we showed that a massive spin particle, i.e. states $|p, \lambda\rangle$ with $\lambda = \pm 1/2$ can be interpolated by the Dirac spinor

$$\psi(x) \equiv u_{\pm}(\vec{p})e^{-ipx} = \langle 0|\Psi(x)|p, \pm\rangle \tag{1}$$

Using explicit expressions show that wave functions $\varepsilon_{\pm}(0)$ (corresponding to the two polarizations) in the rest frame satisfy

$$m\gamma^0 u_{\pm}(0) - m u_{\pm}(0) = 0. \tag{2}$$

Derive what this constraint corresponds to in an arbitrary reference. You should obtain the Dirac equation

$$(i\partial\!\!\!/ - m)\psi(x) = 0, \tag{3}$$

in momentum representation,

Spin 1

Similarly for a vector field $A^{\mu}(x)$ which is used to interpolate for a massive spin 1 particle, i.e. $|p, \lambda\rangle$ with $p^2 = m^2$ and $\lambda = -1, 0, 1$. The wave function

$$\psi^{\mu}(x; p, \lambda) \equiv \varepsilon^{\mu}(p, \lambda)e^{-ipx} = \langle 0|A^{\mu}(x)|p, \lambda\rangle, \tag{4}$$

in the reference frame satisfies

$$\varepsilon^0(\bar{p}, \sigma) = 0. \tag{5}$$

Extending the constraint to an arbitrary frame show that the wave function satisfies

$$(\Box + m^2)\psi^{\mu}(x; p, s) = 0, \quad \partial_{\mu}\psi^{\mu}(x; p, s) = 0. \tag{6}$$

Using two orthogonal projectors

$$\Pi^L_{\mu\nu} = \frac{\partial_{\mu}\partial_{\nu}}{\Box}, \quad \Pi^T_{\mu\nu} = \eta_{\mu\nu} - \Pi^L_{\mu\nu} \tag{7}$$

1. Prove that for $\alpha \neq 0$ constraints (6) are equivalent just to one equation

$$\left[(\Box + m^2)\Pi^T_{\mu\nu} + \alpha\Pi^L_{\mu\nu}\right]\psi^{\nu}(x; p, \sigma) = 0. \tag{8}$$

2. Find α such that this equation becomes local (no inverse powers of $\Box$).

Spin 2

Now consider a massive spin 2 particle, i.e. states $|p,\lambda\rangle$ with $p^2 = m^2$ and $\lambda = -2, -1, 0, 1, 2$. A symmetric rank 2 tensor $h_{\mu\nu}(x)$ can interpolate for those states

$$\psi^{\mu\nu}(x) \equiv \varepsilon^{\mu\nu}(p,\lambda) e^{-ipx} = \langle 0|h^{\mu\nu}(0)|p,\lambda\rangle. \tag{9}$$

1. Find polarizations $\varepsilon^{\mu\nu}(\bar{p},\lambda)$ in the rest frame and identify constraints that they satisfy. Use the fact that $\psi^{\mu\nu}$ can be decomposed as

$$\mathbf{0} + \mathbf{0} + \mathbf{1} + \mathbf{2}, \tag{10}$$

   of the rotation group.

2. Show that those constraints in an arbitrary reference frame can be written as

$$\partial_\mu \psi^{\mu\nu} = 0, \quad \psi^\mu_\mu = 0. \tag{11}$$

3. Prove that the Fierz-Pauli equation

$$\Box\psi_{\mu\nu} - \partial_\sigma\partial_\mu\psi^\sigma_\nu - \partial_\sigma\partial_\nu\psi^\sigma_\mu + \partial_\mu\partial_\nu\psi + \eta_{\mu\nu}(\partial_\lambda\partial_\sigma\psi^{\lambda\sigma} - \Box\psi) + m^2(\psi_{\mu\nu} - \eta_{\mu\nu}\psi) = 0, \tag{12}$$

   with $\psi = \psi^\mu_\mu$ leads to the constraints you above.

4. Projectors constructed in the previous section correspond to spin 0 and 1 representations contained in $\left(\frac{1}{2},\frac{1}{2}\right)$ of ψ^μ. Using $\Pi^{L,T}$ for each index of $\psi_{\mu\nu}$ construct projectors $\Pi^{(2)}$, $\Pi^{(1)}$, $\Pi^{(0_1)}$, $\Pi^{(0_2)}$ corresponding to representations of spin 2, 1 and 0 (two of them) contained in $(2,2)$ of $\psi^{\mu\nu}$. You should get

$$\begin{aligned}
{\Pi^{(2)}}_{\mu\nu,\lambda\sigma} &= \Pi^T_{\mu\lambda}\Pi^T_{\nu\sigma} - \frac{1}{3}\Pi^T_{\mu\nu}\Pi^T_{\lambda\sigma}, \\
{\Pi^{(1)}}_{\mu\nu,\lambda\sigma} &= \Pi^T_{\mu\lambda}\Pi^L_{\nu\sigma} + \Pi^L_{\mu\lambda}\Pi^T_{\nu\sigma}, \\
{\Pi^{(0_1)}}_{\mu\nu,\lambda\sigma} &= \Pi^L_{\mu\nu}\Pi^L_{\lambda\sigma} = \Pi^L_{\mu\lambda}\Pi^L_{\nu\sigma}, \\
{\Pi^{(0_2)}}_{\mu\nu,\lambda\sigma} &= \frac{1}{3}\Pi^T_{\mu\nu}\Pi^T_{\lambda\sigma}.
\end{aligned} \tag{13}$$

   There are also projectors that mix the two spin 0 representations:

$${\Pi^{(0_{12})}}_{\mu\nu,\lambda\sigma} = \frac{1}{\sqrt{3}}\Pi^L_{\mu\nu}\Pi^T_{\lambda\sigma}, \quad {\Pi^{(0_{21})}}_{\mu\nu,\lambda\sigma} = \frac{1}{\sqrt{3}}\Pi^T_{\mu\nu}\Pi^L_{\lambda\sigma}, \tag{14}$$

   satisfying

$${\Pi^{(0_{12})}}_{\mu\nu,\lambda\sigma}{\Pi^{(0_{21})}}^{\lambda\sigma}{}_{\rho\tau} = {\Pi^{(0_1)}}_{\mu\nu,\rho\tau}, \quad {\Pi^{(0_{21})}}_{\mu\nu,\lambda\sigma}{\Pi^{(0_{12})}}^{\lambda\sigma}{}_{\rho\tau} = {\Pi^{(0_2)}}_{\mu\nu,\rho\tau}. \tag{15}$$

5. Write the Fierz-Pauli equation in terms of those projectors. Start by rewriting the mass term as

$$\psi_{\mu\nu} - \eta_{\mu\nu}\psi = (\eta_{\mu\lambda}\eta_{\nu\sigma} - \eta_{\mu\nu}\eta_{\lambda\sigma})\psi^{\lambda\sigma}, \tag{16}$$

and use the following two completeness relations

$$\Pi^{(2)}{}_{\mu\nu,\lambda\sigma} + \Pi^{(1)}{}_{\mu\nu,\lambda\sigma} + \Pi^{(0_1)}{}_{\mu\nu,\lambda\sigma} + \Pi^{(0_2)}{}_{\mu\nu,\lambda\sigma} = \eta_{\mu\lambda}\eta_{\nu\sigma}. \tag{17}$$

and

$$\eta_{\mu\nu} = \Pi^T_{\mu\nu} + \Pi^L_{\mu\nu}. \tag{18}$$