HW12

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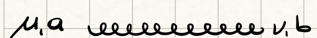
RENORMALIZATION OF YANG-MILLS THEORY

We study the 1-loop renormalization of the Yang-Mills theory with gauge group $SU(N_c)$ and N_f fermions. The Lagrangian is

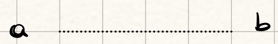
$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + \frac{1}{2\xi}\left(\partial_\mu A^{a\mu}\right)^2 - \partial_\mu \bar{\omega}^a \partial^\mu \omega^a + \bar{\Psi}^i i \not{\partial} \Psi^i \\
& - g f^{abc} \partial_\mu A^a_\nu A^{b\mu} A^{c\nu} - \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} - g f^{abc} \partial_\mu \bar{\omega}^a A^b_\mu \omega^c \\
& + g A^a_\mu \bar{\Psi}^i \gamma^\mu T^a \Psi^i - \frac{\delta_3}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 - \frac{\tilde{\delta}_3}{2\xi}\left(\partial_\mu A^{a\mu}\right)^2 - \tilde{\delta}_2 \partial_\mu \bar{\omega}^a \partial^\mu \omega^a \\
& + \delta_2 \bar{\Psi}^i i \not{\partial} \Psi^i - g\,\delta_{3A} f^{abc} \partial_\mu A^a_\nu A^b_\mu A^c_\nu - \delta_{4A} \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu \\
& - g\,\tilde{\delta}_1 f^{abc} \partial_\mu \bar{\omega}^a A^b_\mu \omega^c + g\,\delta_1 A^a_\mu \bar{\Psi}^i T^a \gamma^\mu \Psi^i
\end{aligned}
$$

The Feynman rules of the theory in $d = 4-\varepsilon$ dimensions are

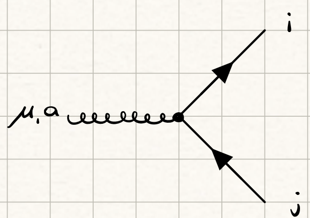
$$= \frac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon}\,\delta^{ij}$$



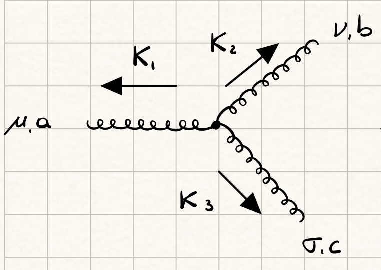
$$= \frac{i}{p^2 + i\varepsilon}\left(-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)\delta^{ab}$$



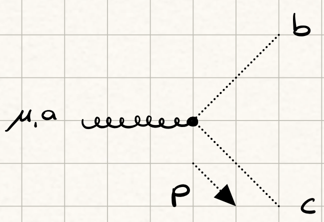
$$= \frac{-i}{p^2 + i\varepsilon}\,\delta^{ab}$$



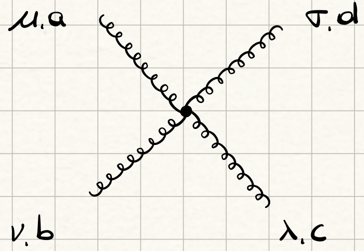
$$= i g \mu^{\varepsilon/2} T^a_{ij} \gamma^\mu$$



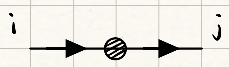
$$= -g\mu^{\varepsilon/2} f^{abc}\left[(K_1 - K_2)_\sigma g_{\mu\nu} + (K_2 - K_3)_\mu g_{\nu\sigma} + (K_3 - K_1)_\nu g_{\mu\sigma}\right]$$



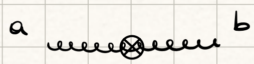
$$= g\mu^{\varepsilon/2} f^{abc} p_\mu$$



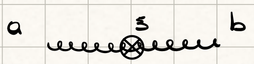
$$= -i g^2 \mu^{\varepsilon} \left[f^{abe} f^{cde} (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}) + f^{ace} f^{bde} (g_{\mu\nu} g_{\lambda\sigma} - g_{\mu\sigma} g_{\nu\lambda}) + f^{ade} f^{bce} (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\nu} g_{\lambda\sigma}) \right]$$



$$= i \delta_2 \not{p} \, \delta^{ij}$$

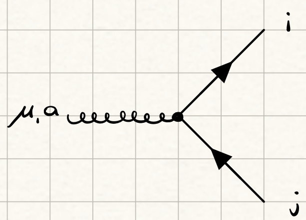


$$= -i \delta_3 (g^{\mu\nu} p^2 - p^{\mu} p^{\nu}) \delta^{ab}$$

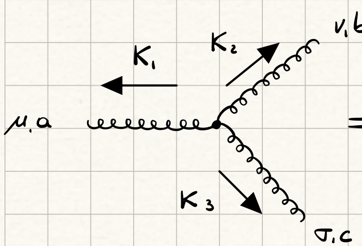


$$= -i \frac{\tilde{\delta}_3}{2\xi} p_{\mu} p_{\nu} \, \delta^{ab}$$

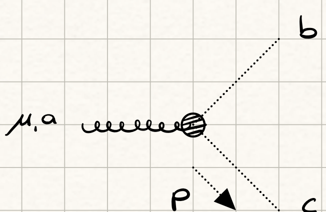
$$= -i \tilde{\delta}_2 p^2 \delta^{ab}$$



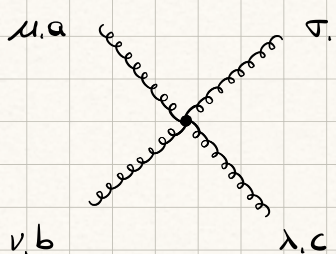
$$= i g \mu^{\varepsilon/2} \delta_1 T^a_{ij} \gamma^{\mu}$$



$$= -g \mu^{\varepsilon/2} \delta_{3A} f^{abc} \left[(k_1 - k_2)_{\sigma} g_{\mu\nu} + (k_2 - k_3)_{\mu} g_{\nu\sigma} + (k_3 - k_1)_{\nu} g_{\mu\sigma} \right]$$



$$= g \mu^{\varepsilon/2} \tilde{\delta}_1 f^{abc} p_{\mu}$$



$$= -i g^2 \mu^{\varepsilon} \delta_{4A} \left[f^{abe} f^{cde} (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}) + f^{ace} f^{bde} (g_{\mu\nu} g_{\lambda\sigma} - g_{\mu\sigma} g_{\nu\lambda}) + f^{ade} f^{bce} (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\nu} g_{\lambda\sigma}) \right]$$

It is convenient to define

$$Z_{(\cdot)} = 1 + \delta_{(\cdot)}$$

From the Ward identities of BRST the following relations between the bare and renormalized coupling follow

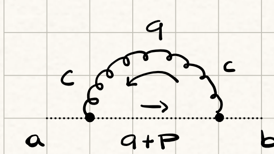
$$\frac{g_0^2}{g^2} = \frac{Z_1^2}{Z_2^2 Z_3} = \frac{Z_{3A}^2}{Z_3^3} = \frac{Z_{4A}}{Z_3^2} = \frac{\tilde{Z}_1^2}{\tilde{Z}_2^2 Z_3}$$

which enstablish the equality of the 3-gluon, 4-gluon, fermion-gluon, ghost-gluon couplings to all orders.

PURE YANG-MILLS THEORY

Let us start studying the theory for $N_f = 0$. We are interested in the compution of the 1-loop beta function for the gauge coupling.
This can be done computing the wave function renormalization $\tilde{Z}_1$, $\tilde{Z}_2$ and Z_3 and then using the relation above. We will work with $\xi = 1$.

GHOST WAVE FUNCTION RENORMALIZATION



$$= g^2 \mu^\varepsilon f^{cak} f^{ckb} \int \frac{d^d q}{(2\pi)^d} \frac{-i\,(p+q)_\mu p_\nu}{(q+p)^2 + i\varepsilon} \frac{i}{q^2 + i\varepsilon} (-g^{\mu\nu})$$

$$= -g^2 \mu^\varepsilon \mathrm{Tr}\{T^a_{Adj} T^b_{Adj}\} \int \frac{d^d q}{(2\pi)^d} \frac{(p+q)_\mu p_\nu}{(q+p)^2 + i\varepsilon} \frac{1}{q^2 + i\varepsilon} (-g^{\mu\nu})$$

$$= g^2 \mu^\varepsilon N_c \delta^{ab} \int \frac{d^d q}{(2\pi)^d} \frac{p\cdot(p+q)}{(q+p)^2 + i\varepsilon} \frac{1}{q^2 + i\varepsilon}$$

$$= g^2 \mu^\varepsilon N_c \delta^{ab} \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{p\cdot(p+q)}{(q^2 + (p^2 + 2p\cdot q)x + i\varepsilon)^2} =$$

$$= g^2 \mu^\varepsilon N_c \delta^{ab} \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{(1-x)\, p^2}{(q^2 + p^2 x(1-x) + i\varepsilon)^2} =$$

$$= i \frac{g^2}{(4\pi)^2} N_c \delta^{ab} p^2 \frac{2}{\varepsilon} \int_0^1 dx\,(1-x) + O(\varepsilon^0) =$$

$$= \frac{i}{\varepsilon} \frac{g^2}{(4\pi)^2} N_c \delta^{ab} p^2 + O(\varepsilon^0)$$

$$= -i\, \tilde{\delta}_2^{(1)} p^2 \delta^{ab}$$

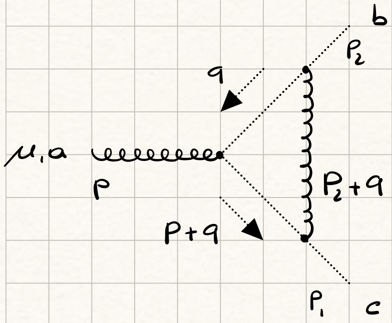
The divergence is removed with the following choice for the counterterm in the minimal subtraction scheme

$$\tilde{\delta}_2^{(1)} = \frac{1}{\varepsilon} \frac{g^2}{(4\pi)^2} N_c$$

Thus, we find the 1-loop ghost wave function renormalization

$$\tilde{Z}_2 = 1 + \frac{N_c}{\varepsilon}\frac{g^2}{16\pi^2} + O(g^4)$$

GHOST GLUON INTERACTION VERTEX



$$= g^3 \mu^{3\varepsilon/2} f^{aks} f^{rsc} f^{rbk} \int \frac{d^dq}{(2\pi)^d} \frac{-i(p+q)_\mu}{(p+q)^2+i\varepsilon} \frac{-i P_1\cdot q}{(P_2+q)^2+i\varepsilon}\frac{-i}{q^2+i\varepsilon} =$$

$$= i g^3\mu^{3\varepsilon/2}\, i\, \mathrm{Tr}\{T^a_{Adj}T^b_{Adj}T^c_{Adj}\} \int \frac{d^dq}{(2\pi)^d}\frac{(p+q)_\mu}{(p+q)^2+i\varepsilon}\frac{P_1\cdot q}{(P_2+q)^2+i\varepsilon}\frac{1}{q^2+i\varepsilon} =$$

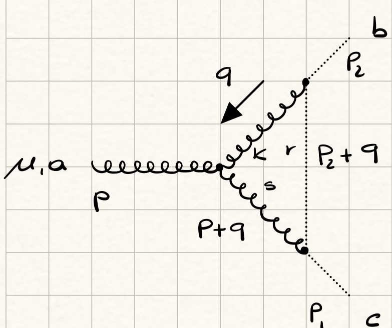
$$= -i g^3 \mu^{3\varepsilon/2}\frac{N_c}{2} f^{abc}\int \frac{d^dq}{(2\pi)^d}\frac{(p+q)_\mu}{(p+q)^2+i\varepsilon}\frac{P_1\cdot q}{(q^2+i\varepsilon)^2} + O(\varepsilon^0) =$$

$$= -\frac{i}{2} g^3\mu^{3\varepsilon/2} N_c f^{abc}\, 2\int_0^1 dx\,(1-x)\int \frac{d^dq}{(2\pi)^d}\frac{(p+q)_\mu\, P_1\cdot q}{(q^2+(p^2+2p\cdot q)x+i\varepsilon)^3} + O(\varepsilon^0)$$

$$= -\frac{i}{2} g^3\mu^{3\varepsilon/2}N_c f^{abc}\,2\int_0^1 dx\,(1-x)\int\frac{d^dq}{(2\pi)^d}\frac{q_\mu q_\rho}{(q^2+x(1-x)p^2+i\varepsilon)^3} P_1^\rho + O(\varepsilon^0)$$

$$= -i g^3\mu^{\varepsilon/2} N_c f^{abc}\int_0^1 dx\,(1-x)\frac{1}{(4\pi)^2}\frac{i}{2}\frac{1}{2}\frac{2}{\varepsilon} P_\mu + O(\varepsilon^0) =$$

$$= +\mu^{\varepsilon/2} g \frac{N_c}{4}\frac{1}{\varepsilon}\frac{g^2}{16\pi^2} f^{abc} P_\mu + O(\varepsilon^0)$$



$$= -g^3\mu^{3\varepsilon/2} f^{aks} f^{src} f^{kbr} \times$$

$$\int \frac{d^dq}{(2\pi)^d}\left[-(p-q)_\sigma g_{\mu\nu} + (-p-2q)_\mu g_{\nu\sigma} + (2p+q)_\nu g_{\mu\sigma}\right]$$

The div. part does not depend on P_2 (becau the derivative is finite) therefore, we isolate the $P_2=0$ contribution.

$$\times (-i)\frac{-P_1^\sigma (P_2+q)^\nu}{(P_2+q)^2+i\varepsilon}\frac{i}{(p+q)^2+i\varepsilon}\frac{i}{q^2+i\varepsilon} =$$

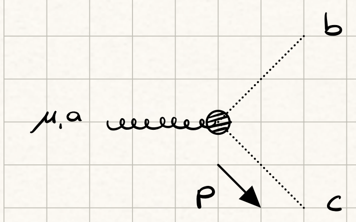
↓

$$= -i g^3\mu^{3\varepsilon/2}\frac{N_c}{2} f^{abc}\int\frac{d^dq}{(2\pi)^d}\frac{P_1\cdot q\, q_\mu - 2q_\mu(P_1\cdot q) + P_{1\mu} q^2}{((p+q)^2+i\varepsilon)(q^2+i\varepsilon)^2} + O(\varepsilon^0) =$$

$$= -i\, g^3 \mu^{3\varepsilon/2} N_c f^{abc} \int_0^1 dx\,(1-x) \int \frac{d^dq}{(2\pi)^d} \frac{P_{1\mu} q^2 - q_\mu (P_1\cdot q)}{(q^2 + x(1-x)P^2 + i\varepsilon)^3} + O(\varepsilon^0) =$$

$$= -i\, g^3 \mu^{\varepsilon/2} N_c f^{abc} \int_0^1 dx\,(1-x) \frac{i}{16\pi^2} \left(P_{1\mu} - \frac{1}{4} P_{1\mu}\right) \frac{2}{\varepsilon} + O(\varepsilon^0) =$$

$$= \mu^{\varepsilon/2} f^{abc} \frac{3}{4} N_c \frac{g^3}{16\pi^2} \frac{1}{\varepsilon} + O(\varepsilon^0)$$



$$= g\mu^{\varepsilon/2} \tilde{\delta}_1^{(1)} f^{abc} P_{1\mu}$$

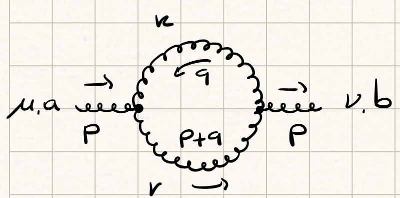
At this point, we can identify the counterterm in the MS scheme

$$\tilde{\delta}_1^{(1)} = \left(-\frac{1}{4} - \frac{3}{4}\right) N_c \frac{g^2}{16\pi^2} \frac{1}{\varepsilon}$$

which gives

$$\tilde{Z}_1 = 1 - N_c \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} + O(g^4)$$

GLUON WAVEFUNCTION RENORMALIZATION



$$= -g^2 \mu^\varepsilon f^{akr} f^{bkr} \frac{1}{2} \int \frac{d^dq}{(2\pi)^d} \frac{i}{(p+q)^2 + i\varepsilon} \frac{i}{q^2 + i\varepsilon}$$

$$[(q-p)_\sigma g_{\mu\rho} + (-2q-p)_\mu g_{\rho\sigma} + (2p+q)_\rho g_{\mu\sigma}][(q-p)^\sigma g_\nu{}^\rho + (-2q-p)_\nu g^{\rho\sigma} + (2p+q)^\rho g_\nu{}^\sigma]$$

$$= g^2 \mu^\varepsilon N_c \delta^{ab} \frac{1}{2} \int_0^1 dx \int \frac{d^dq}{(2\pi)^d} \frac{2q^2 g_{\mu\nu} + 2p\cdot q\, g_{\mu\nu} + 5p^2 g_{\mu\nu} + 10 q_\mu q_\nu + 5p_\mu q_\nu + 5p_\nu q_\mu - 2p_\mu p_\nu}{((q+xp)^2 + x(1-x)p^2 + i\varepsilon)^2}$$

$$= g^2 \mu^\varepsilon N_c \delta^{ab} \frac{1}{2} \int_0^1 dx$$

$$\int \frac{d^dq}{(2\pi)^d} \frac{(2q^2 + 2x^2p^2 - 2xp^2 + 5p^2) g_{\mu\nu} + 10 q_\mu q_\nu + 10x^2 p_\mu p_\nu - 10x\, p_\mu p_\nu - 2 p_\mu p_\nu}{(q^2 + x(1-x)p^2 + i\varepsilon)^2} =$$

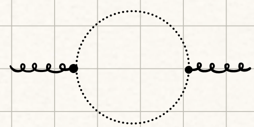
$$= g^2 \frac{N_c}{2} \delta^{ab} \frac{i}{16\pi^2} \int_0^1 dx \Big[4(-x(1-x))P^2 \frac{2}{\varepsilon} g_{\mu\nu} + \frac{2}{\varepsilon}(-2x(1-x)P^2 + 5P^2) g_{\mu\nu} +$$

$$+ 5(-x(1-x))P^2 \frac{2}{\varepsilon} g_{\mu\nu} - \frac{2}{\varepsilon} 10x(1-x) P_\mu P_\nu - \frac{2}{\varepsilon} 2 P_\mu P_\nu \Big] + O(\varepsilon^0)$$

$$= g^2 \frac{N_c}{2} \delta^{ab} \frac{i}{16\pi^2} \frac{2}{\varepsilon} \int_0^1 dx\, ((-4-2-5)x(1-x) + 5P^2) g_{\mu\nu} +$$

$$-(10x(1-x)+2) P_\mu P_\nu + O(\varepsilon^0) =$$

$$= g^2 N_c \delta^{ab} \frac{i}{16\pi^2} \left(\frac{19}{12} P^2 g_{\mu\nu} - \frac{11}{6} P_\mu P_\nu \right) \frac{2}{\varepsilon} + O(\varepsilon^0)$$

μ, a  ν, b

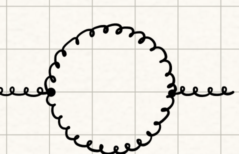
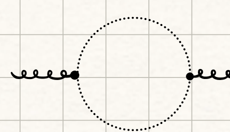
$$= (-) - g^2 \mu^\varepsilon f^{aks} f^{bsk} \int \frac{d^d q}{(2\pi)^d} \frac{(p+q)_\mu}{(p+q)^2 + i\varepsilon} \frac{q_\nu}{q^2 + i\varepsilon} =$$

$$= - g^2 \mu^\varepsilon N_c \delta^{ab} \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{q_\mu q_\nu - x(1-x) P_\mu P_\nu}{(q^2 + x(1-x)P^2 + i\varepsilon)^2} =$$

$$= - g^2 N_c \frac{i}{16\pi^2} \delta^{ab} \frac{2}{\varepsilon} \int_0^1 dx \Big[(-x(1-x)P^2) \frac{1}{2} g_{\mu\nu} +$$

$$- x(1-x) P_\mu P_\nu \Big] + O(\varepsilon^0) =$$

$$= g^2 N_c \delta^{ab} \frac{i}{16\pi^2} \frac{2}{\varepsilon} \left(+ \frac{1}{12} P^2 g_{\mu\nu} + \frac{1}{6} P_\mu P_\nu \right) + O(\varepsilon^0)$$

μ, a  ν, b + μ, a  ν, b

$$= i \frac{10}{3} N_c \frac{g^2}{16\pi^2} (P^2 g_{\mu\nu} - P_\mu P_\nu) \frac{1}{\varepsilon} \delta^{ab} + O(\varepsilon^0)$$

a  b

$$= -i \delta_3^{(1)} (g^{\mu\nu} P^2 - P^\mu P^\nu) \delta^{ab}$$

$$\delta_3^{(1)} = N_c \frac{5}{3} \frac{g^2}{16\pi^2} \frac{2}{\varepsilon}$$

Therefore, we find the gluon wavefunction renormalization

$$Z_3 = 1 + \frac{10}{3} N_c \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} + O(g^4)$$

Overall we have

$$\tilde{Z}_2 = 1 + \frac{N_c}{\varepsilon}\frac{g^2}{16\pi^2} + O(g^4)$$

$$\tilde{Z}_1 = 1 - N_c \frac{g^2}{16\pi^2}\frac{1}{\varepsilon} + O(g^4)$$

$$Z_3 = 1 + \frac{10}{3} N_c \frac{g^2}{16\pi^2}\frac{1}{\varepsilon} + O(g^4)$$

At this point we can compute the beta function

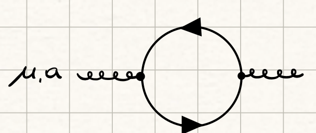
$$\alpha_0 = \mu^\varepsilon \alpha \frac{\tilde{Z}_1^2}{\tilde{Z}_2^2 Z_3} = \mu^\varepsilon \alpha \left(1 + \left(-2N_c - 2N_c + \frac{10}{3}N_c\right)\frac{\alpha}{4\pi}\frac{1}{\varepsilon} + O(\alpha^2)\right)$$
$$= \mu^\varepsilon \alpha \left(1 - \frac{11}{3} N_c \frac{\alpha}{2\pi}\frac{1}{\varepsilon} + O(\alpha^2)\right)$$

The β-function can be computed directly from the residue of the $1/\varepsilon$ pole in the bare coupling expansion. We find

$$\beta(\alpha) = -\frac{11}{3} N_c \frac{\alpha^2}{2\pi} + O(\alpha^3)$$

ADDING FERMIONS

Let us now consider the problem with N_f fermions in a given representation R of the gauge group. The only diagram we have to study is the contribution to the vacuum polarization which differs from the abelian case only for a group factor (I_R is the Dynkin index of the representation)

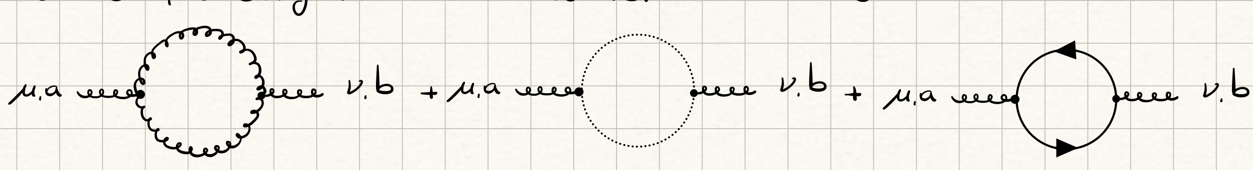


$$= (-) g^2 \mu^\varepsilon \, Tr\{T_R^a T_R^b\} N_f \int \frac{d^d q}{(2\pi)^d} \frac{Tr\{\gamma^\mu(\not{q}+m)\gamma^\nu(\not{p}+\not{q}+m)\}}{((p+q)^2 - m^2 + i\varepsilon)(q^2 - m^2 + i\varepsilon)}$$

$$= -i \frac{8}{3}\frac{g^2}{16\pi^2} I_R N_f (p^2 g^{\mu\nu} - p^\mu p^\nu)\frac{1}{\varepsilon}\delta^{ab} + O(\varepsilon^0)$$

From the QED computation ↗

Therefore, adding this contribution to the other two found before, we get



$$= i(p^2 g^{\mu\nu} - p^\mu p^\nu)\delta^{ab}\left(\frac{10}{3}N_c - \frac{8}{3} I_R N_f\right)\frac{g^2}{16\pi^2}\frac{1}{\varepsilon} + O(\varepsilon^0)$$

and thus

$$Z_3 = 1 + \left(\frac{10}{3}N_c - \frac{8}{3} I_R N_f\right)\frac{g^2}{16\pi^2}\frac{1}{\varepsilon} + O(\varepsilon^0)$$

Overall, we found the following wave function renormalization constants

$$\tilde{Z}_2 = 1 + N_c \frac{\alpha}{4\pi} \frac{1}{\varepsilon} + O(g^4)$$

$$\tilde{Z}_1 = 1 - N_c \frac{\alpha}{4\pi} \frac{1}{\varepsilon} + O(g^4)$$

$$Z_3 = 1 + \left(\frac{10}{3} N_c - \frac{8}{3} I_R N_f \right) \frac{\alpha}{4\pi} \frac{1}{\varepsilon} + O(\varepsilon^0)$$

At this point, we can compute the bare coupling

$$\alpha_0 = \alpha \mu^\varepsilon \frac{\tilde{Z}_1^2}{\tilde{Z}_2^2 Z_3} = \alpha \mu^\varepsilon \Big(1 + \Big(-2N_c - 2N_c - \frac{10}{3} N_c + \frac{8}{3} I_R N_f \Big) \frac{\alpha}{4\pi} \frac{1}{\varepsilon} + O(g^4)$$

$$= \alpha \mu^\varepsilon \left(1 + \left(\frac{8}{3} I_R N_f - \frac{22}{3} N_c \right) \frac{\alpha}{4\pi} \frac{1}{\varepsilon} + O(g^4)\right)$$

The β-function is

$$\beta(\alpha) = \left(\frac{4}{3} I_R N_f - \frac{11}{3} N_c \right) \frac{\alpha^2}{2\pi} + O(\alpha^3)$$

$$\alpha = \frac{g^2}{4\pi} \qquad g = \sqrt{4\pi\alpha}$$

$$\beta_g = \frac{dg}{d\alpha} \beta_\alpha = \frac{1}{2} \sqrt{\frac{4\pi}{\alpha}} \beta_\alpha = \frac{2\pi}{g} \left(\frac{4}{3} I_R N_f - \frac{11}{3} N_c \right) \frac{g^4}{32\pi^3}$$

$$\beta_g = \left(\frac{4}{3} I_R N_f - \frac{11}{3} N_c \right) \frac{g^3}{16\pi^2}$$

COMPARISON WITH QED

The beta function for a U(1) gauge theory with N_f fermions with equal charge is (the MS scheme is used)

$$\beta^{MS}_{U(1)}(\alpha) = \frac{4}{3} N_f \frac{\alpha^2}{2\pi} + O(\alpha^3)$$

At the same time the β-function for an SU(N) gauge theory with N_f fermions is given by

$$\beta^{MS}_{SU(N_c)}(\alpha) = \left(\frac{4}{3} I_R N_f - \frac{11}{3} N_c \right) \frac{\alpha^2}{2\pi} + O(\alpha^3)$$

We notice that in the matter contribution of the β-function it appears the the Dynkin index of the fermion representation which is one in the abelian case. Crucially, for Yang-Mills theories there is an additional negative contribution which makes all the β-function negative for (asymptotic freedom)

$$N_c > \frac{4}{11} N_c I_R$$