HW 11

ONE LOOP STRUCTURE OF QED

Let us consider the QED Lagrangian

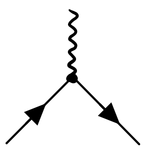
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - e\mu^{\varepsilon/2} A_\mu \bar{\psi}\gamma^\mu\psi$$

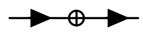
The Feynman rules are

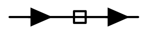
Counterterms

 $\frac{i}{p^2+i\varepsilon}(-g_{\mu\nu})$

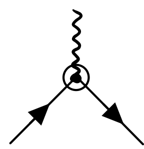
 $\frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$

 $= -ie\mu^{\varepsilon/2}\gamma^\mu$

 $i\delta_\psi \not{p}$

 $-i\delta_m$

 $-i\delta_A(p^2 g^{\mu\nu} - p^\mu p^\nu)$

 $= -i\mu^{\varepsilon/2}\delta_e\gamma^\mu$

Let us identify the divergent contributions to correlation functions.
We recall the formula

$$L = I_\psi + I_A - V + 1$$

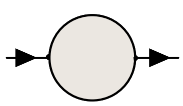
where

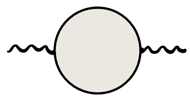
$$V = 2I_A + E_A = I_\psi + \frac{1}{2}E_\psi$$

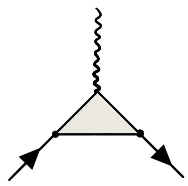
The superficial degree of divergence of a generic diagram is

$$\begin{aligned}\delta(G) &= 4L - 2I_A - I_\psi = \\ &= 4I_\psi + 4I_A - 4V + 4 - 2I_A - I_\psi = \\ &= 4 + 3I_\psi + 2I_A - 4V = 4 - \frac{3}{2}E_\psi - E_A\end{aligned}$$

As a consequence of Lorentz invariance and C symmetry, the 1PI correlation functions we need to renormalize are

$E_\psi = 2 \quad E_A = 0$  FERMION SELF ENERGY

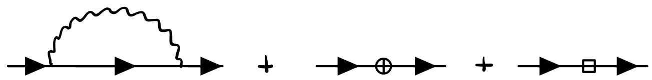
$E_\psi = 0 \quad E_A = 2$  VACUUM POLARIZATION

$E_\psi = 2 \quad E_A = 1$  VERTEX FUNCTION

Let us compute the 1-loop contribution

FERMION SELF ENERGY

Let us compute the divergent contribution of the 1-loop diagram

$-i\,\Sigma^{(1)}_\psi(\not p) =$  $=$

$$= \mu^\varepsilon e^2 \int \frac{d^dq}{(2\pi)^d} \frac{-\gamma^\mu(\not q + m)\gamma_\mu}{((p-q)^2+i\varepsilon)(q^2-m^2+i\varepsilon)} + i(\delta_\psi \not p - \delta_m) =$$

$$= \mu^\varepsilon e^2 \int_0^1 dx \int \frac{d^dq}{(2\pi)^d} \frac{-\gamma^\mu(\not q + m)\gamma_\mu}{((q^2-m^2)+(p^2-2p\cdot q+m^2)x + i\varepsilon)^2} + i(\delta_\psi \not p - \delta_m) =$$

$$= \mu^\varepsilon e^2 \int_0^1 dx \int \frac{d^dq}{(2\pi)^d} \frac{-\gamma^\mu(\not q + m)\gamma_\mu}{((q-px)^2 + p^2x(1-x) + m^2(x-1) + i\varepsilon)^2} + i(\delta_\psi \not p - \delta_m) =$$

$$= \mu^\varepsilon e^2 \int_0^1 dx \int \frac{d^dq}{(2\pi)^d} \frac{-\gamma^\mu(\not q + \not p x + m)\gamma_\mu}{(q^2-\Delta+i\varepsilon)^2} + i(\delta_\psi \not p - \delta_m) =$$

$$\Delta = -x(1-x)p^2 + m^2(1-x)$$

$$= \mu^\varepsilon e^2 \int_0^1 dx \int \frac{d^dq}{(2\pi)^d} \frac{+(d-2)\not p x - dm}{(q^2-\Delta+i\varepsilon)^2} + i(\delta_\psi \not p - \delta_m) =$$

$$= e^2\, i \int_0^1 dx \left[+(d-2)\not p x - dm\right] \frac{\mu^\varepsilon}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \Delta^{\frac{d}{2}-2} + i(\delta_\psi \not p - \delta_m) =$$

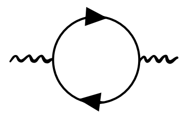
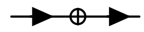
$$= i\frac{e^2}{(4\pi)^2} \int_0^1 dx\, (+2\not p x - 4m)\left(\frac{2}{\varepsilon} + O(\varepsilon^0)\right) + i(\delta_\psi \not p - \delta_m) =$$

$$= i\frac{e^2}{(4\pi)^2}\frac{2}{\varepsilon}(+\not p - 4m) + i(\delta_\psi \not p - \delta_m) + O(\varepsilon^0)$$

At this point, we can identify the counterterms in the minimal subtraction scheme

$$\delta^{(1)}_\psi = -\frac{2}{\varepsilon}\frac{e^2}{(4\pi)^2} \qquad \delta^{(1)}_m = -\frac{8}{\varepsilon}\frac{me^2}{(4\pi)^2}$$

VACUUM POLARIZATION

$$i\Pi^{\mu\nu}_{(1)}(q) = i\Pi_{(1)}(P^2)(P^2 g^{\mu\nu} - P^\mu P^\nu) =$$  $+$  $=$

$$= -\mu^\varepsilon e^2 \int \frac{d^dq}{(2\pi)^d} \frac{\mathrm{Tr}\{(\not{q}+\not{P}+m)\gamma^\mu(\not{q}+m)\gamma^\nu\}}{((P+q)^2-m^2+i\varepsilon)(q^2-m^2+i\varepsilon)} - i\delta_A(P^2 g^{\mu\nu} - P^\mu P^\nu) =$$

$$= -\mu^\varepsilon e^2 \int_0^1 dx \int \frac{d^dq}{(2\pi)^d} \frac{\mathrm{Tr}\{(\not{q}+\not{P}+m)\gamma^\mu(\not{q}+m)\gamma^\nu\}}{(q^2-m^2+(P^2+2P\cdot q)x+i\varepsilon)^2} - i\delta_A(P^2 g^{\mu\nu} - P^\mu P^\nu) =$$

$$= -\mu^\varepsilon e^2 \int_0^1 dx \int \frac{d^dq}{(2\pi)^d} \frac{\mathrm{Tr}\{(\not{q}+\not{P}+m)\gamma^\mu(\not{q}+m)\gamma^\nu\}}{((q+Px)^2 + P^2x(1-x) - m^2+i\varepsilon)^2}$$

$$- i\delta_A(P^2 g^{\mu\nu} - P^\mu P^\nu) =$$

$$= -\mu^\varepsilon e^2 \int_0^1 dx \int \frac{d^dq}{(2\pi)^d} \frac{\mathrm{Tr}\{(\not{q}+(1-x)\not{P}+m)\gamma^\mu(\not{q}-x\not{P}+m)\gamma^\nu\}}{(q^2-(-P^2x(1-x)+m^2)+i\varepsilon)^2}$$

$$- i\delta_A(P^2 g^{\mu\nu} - P^\mu P^\nu) =$$

$$= -\mu^\varepsilon e^2 \int_0^1 dx \int \frac{d^dq}{(2\pi)^d} \frac{\mathrm{Tr}\{(\not{q}+(1-x)\not{P}+m)\gamma^\mu(\not{q}-x\not{P}+m)\gamma^\nu\}}{(q^2-\Delta+i\varepsilon)^2}$$

$$- i\delta_A(P^2 g^{\mu\nu} - P^\mu P^\nu)$$

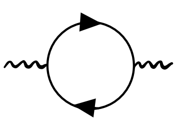
Let us consider the numerator

$$\mathrm{Tr}\{(\not{q}+(1-x)\not{P}+m)\gamma^\mu(\not{q}-x\not{P}+m)\gamma^\nu\} =$$

$$= \mathrm{Tr}\{\not{q}\gamma^\mu\not{q}\gamma^\nu\} - x(1-x)\,\mathrm{Tr}\{\not{P}\gamma^\mu\not{P}\gamma^\nu\} + m^2\,\mathrm{Tr}\{\gamma^\mu\gamma^\nu\} + \text{linear terms in } q$$

$$= 4(2q^\mu q^\nu - g^{\mu\nu}q^2) - x(1-x)4(2P^\mu P^\nu - P^2 g^{\mu\nu}) + 4m^2 g^{\mu\nu} + \text{linear terms in } q$$

We notice that the linear terms in q do not contribute because the denominator is an even function. Overall, we find the diagram to be

 $$= -i4\mu^\varepsilon \frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx \Big[\frac{-1}{d}(2-d) g^{\mu\nu} \frac{d}{2} \Gamma(1-d/2)\Delta^{\frac{d}{2}-1} +$$

$$-x(1-x)(2P^\mu P^\nu - P^2 g^{\mu\nu})\Gamma(2-d/2)\Delta^{\frac{d}{2}-2} + m^2 g^{\mu\nu}\Gamma(2-d/2)\Delta^{\frac{d}{2}-2}\Big] =$$

$$= -i4\mu^\varepsilon \frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx \,[(P^2x(1-x)g^{\mu\nu} - m^2 g^{\mu\nu}) - x(1-x)(2P^\mu P^\nu - P^2 g^{\mu\nu})$$

$$+ m^2 g^{\mu\nu})]\,\Gamma(2-\tfrac{d}{2})\Delta^{\frac{d}{2}-2} =$$

$$= -4i \frac{e^2}{(4\pi)^2} \int_0^1 dx \left[p^2 x(1-x) g^{\mu\nu} - x(1-x)\left(2p^\mu p^\nu - p^2 g^{\mu\nu}\right) \right] \left(\frac{2}{\varepsilon} + O(\varepsilon^0) \right)$$

$$= -i \frac{e^2}{(4\pi)^2} \frac{8}{\varepsilon} \left(\frac{1}{6} p^2 g^{\mu\nu} - \frac{1}{3} p^\mu p^\nu + \frac{1}{6} p^2 g^{\mu\nu} \right) + O(\varepsilon^0) =$$

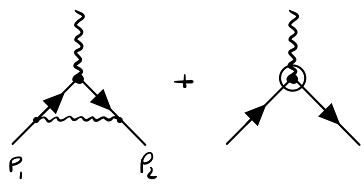
$$= -i \frac{8}{3} \frac{e^2}{(4\pi)^2} \frac{1}{\varepsilon} \left(p^2 g^{\mu\nu} - q^\mu q^\nu \right) + O(\varepsilon^0)$$

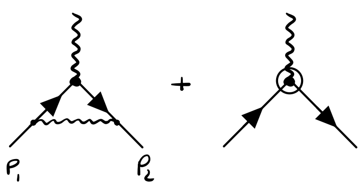
At this point we can identify the counterterm

$$\delta_A^{(1)} = -\frac{8}{3} \frac{e^2}{(4\pi)^2} \frac{1}{\varepsilon}$$

VERTEX FUNCTION

$$-ie\mu^{\varepsilon/2} \Gamma^\mu_{(1)}(p_1, p_2) =$$





$$= -\mu^{\frac{3}{2}\varepsilon} e^3 \int \frac{d^dq}{(2\pi)^d} \frac{1}{q^2+i\varepsilon} \gamma^\nu \frac{(\not{p}_2 + \not{q} + m)}{(p_2+q)^2 - m^2 + i\varepsilon} \gamma^\mu \frac{(\not{p}_1 + \not{q} + m)}{(p_1+q)^2 - m^2 + i\varepsilon} \gamma_\nu$$

$$- i\mu^{\varepsilon/2} \delta_e^{(1)} \gamma^\mu$$

We notice that the diagram has a superficial degree of divergence equal to zero in $d=4$. Therefore, if we were to differentiate in the external momenta or in the fermion mass, we would find a convergent result. This means that the divergence does not depend on p_1, p_2 or m and we can extract it computing the vertex function at zero momentum

$$\Gamma^\mu_{(1)}(0,0) = -\mu^{\frac{3}{2}\varepsilon} e^3 \int \frac{d^dq}{(2\pi)^d} \frac{\gamma^\nu (\not{q}+m) \gamma^\mu (\not{q}+m) \gamma_\nu}{(q^2+i\varepsilon)(q^2-m^2+i\varepsilon)^2} - i\mu^{\varepsilon/2} \delta_e^{(1)} \gamma^\mu =$$

$$= -\mu^{\frac{3}{2}\varepsilon} 2 e^3 \int_0^1 dx\, x \int \frac{d^dq}{(2\pi)^d} \frac{\gamma^\nu \not{q} \gamma^\mu \not{q} \gamma_\nu + m^2 \gamma^\nu \gamma^\mu \gamma_\nu}{(q^2 - m^2 x + i\varepsilon)^3} - i\mu^{\varepsilon/2} \delta_e^{(1)} \gamma^\mu =$$

Let us consider the numerator

$$\gamma^\nu \not{q} \gamma^\mu \not{q} \gamma_\nu + m^2 \gamma^\nu \gamma^\mu \gamma_\nu =$$

$$= \gamma^\nu \{\not{q}, \gamma^\mu\} \not{q} \gamma_\nu - \gamma^\nu \gamma^\mu \not{q} \not{q} \gamma_\nu - m^2 (d-2) \gamma^\mu =$$

$$= 2 q^\mu \gamma^\nu \not{q} \gamma_\nu - q^2 \gamma^\nu \gamma^\mu \gamma_\nu - m^2 (d-2) \gamma^\mu =$$

$$= -2(d-2) q^\mu \not{q} + (d-2) q^2 \gamma^\mu - m^2 (d-2) \gamma^\mu =$$

Let us compute the divergent contribution

$$\Gamma^{\mu}_{(1)}(0,0) = -(d-2)\,\mu^{\frac{3}{2}\varepsilon} 2e^3 \int_0^1 dx\, x \int \frac{d^dq}{(2\pi)^d} \frac{-2q^{\mu}\not{q} + q^2\gamma^{\mu}}{(q^2-m^2x+i\varepsilon)^3} + O(\varepsilon^0) \quad -i\mu^{\varepsilon/2}\delta_e^{(1)}\gamma^{\mu}$$

$$= -(d-2)\mu^{\frac{3}{2}\varepsilon} 2e^3\gamma^{\mu} \int_0^1 dx\, x \left(-\frac{2}{d}+1\right) \int \frac{d^dq}{(2\pi)^d} \frac{q^2}{(q^2-m^2x+i\varepsilon)^3} + O(\varepsilon^0) + c.t.$$

$$= -\mu^{\frac{3}{2}\varepsilon} 2e^3\gamma^{\mu}\frac{(d-2)^2}{d}\int_0^1 dx\, x \frac{i}{(4\pi)^{d/2}}\frac{d}{2}\frac{1}{2}\Gamma\left(2-\frac{d}{2}\right)(m^2x)^{\frac{d}{2}-2} + O(\varepsilon^0) + c.t.$$

$$= -\mu^{\varepsilon/2}\frac{i}{2}\frac{e^3}{(4\pi)^2}\gamma^{\mu}(d-2)^2\,\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\, x + O(\varepsilon^0) - i\mu^{\varepsilon/2}\delta_e^{(1)}\gamma^{\mu}$$

$$= -ie\mu^{\varepsilon/2}\gamma^{\mu}\left(\frac{2}{\varepsilon}\frac{e^2}{(4\pi)^2} + O(\varepsilon^0)\right) - i\mu^{\varepsilon/2}\delta_e^{(1)}\gamma^{\mu}$$

We can identify the counterterm in the minimal subtraction scheme

$$\delta_e^{(1)} = -\frac{2}{\varepsilon}\frac{e^3}{(4\pi)^2}$$

Let us collect the results for the counterterms

$$\delta_{\psi}^{(1)} = -\frac{2}{\varepsilon}\frac{e^2}{(4\pi)^2} \qquad \delta_m^{(1)} = -\frac{8}{\varepsilon}\frac{me^2}{(4\pi)^2}$$

$$\delta_A^{(1)} = -\frac{8}{3}\frac{e^2}{(4\pi)^2}\frac{1}{\varepsilon} \qquad \delta_e^{(1)} = -\frac{2}{\varepsilon}\frac{e^3}{(4\pi)^2}$$

The relation between the renormalized and the bare charge is given by

$$e_0 = \mu^{\varepsilon/2} Z_A^{-1/2} Z_{\psi}^{-1}(e + \delta_e)$$

At one loop we have

$$e_0 = \mu^{\varepsilon/2}\left(1 - \frac{1}{2}\delta_A^{(1)}\right)\left(1 - \delta_{\psi}^{(1)}\right)\left(e + \delta_e^{(1)}\right) + \ldots =$$

$$= \mu^{\varepsilon/2}\left(e + e\left(-\frac{1}{2}\delta_A^{(1)} - \delta_{\psi}^{(1)} + \frac{1}{e}\delta_e^{(1)}\right) + \ldots\right)$$

Now we notice that
$$\delta_{\psi}^{(1)} - \frac{1}{e}\delta_e^{(1)} = 0$$

This cancellation comes from the Ward identity and teaches us that the charge is only renormalized by the vacuum polarization diagram.

$$e_0 = \mu^{\varepsilon/2}\left(e - \frac{e}{2}\delta_A^{(1)} + \ldots\right)$$

Finally, we can derive the beta function for the electric charge.

$$\mu \frac{d}{d\mu} e_0 = 0 = \mu^{\varepsilon/2} \left(\frac{\varepsilon}{2} e - \frac{\varepsilon}{2} \frac{e}{2} \delta_A^{(1)} + \beta_e(\varepsilon) - \frac{3}{2} \delta_A^{(1)} \beta_e(\varepsilon) + \dots \right)$$

Using $\beta_e(\varepsilon) = -\frac{\varepsilon}{2} e + \beta_e^{(1)} + \dots$

we find

$$\beta_e^{(1)} - \frac{e}{4}(\varepsilon \delta_A^{(1)}) + \frac{3}{4} e (\varepsilon \delta_A^{(1)}) = 0 \qquad \beta_e^{(1)} = -\frac{1}{2} e (\varepsilon \delta_A^{(1)})$$

and thus the 1-loop β-function is

$$\beta_e^{(1)} = \frac{4}{3} \frac{e^3}{(4\pi)^2}$$

It is convenient to write it in terms of $\alpha = \frac{e^2}{4\pi}$

$$\beta(\alpha) = \mu \frac{d\alpha}{d\mu} = \frac{e}{2\pi} \beta_e = \frac{4}{3} \frac{\alpha^2}{2\pi} + O(\alpha^3)$$