▨ CRITICAL EXPONENTS HW 9

$$\mathcal{L} = \frac{1}{2}(\partial\phi_i)^2 + \frac{\lambda}{4!}\mu^{\varepsilon}(\phi_i\phi_i)^2 + \text{counterterms}$$

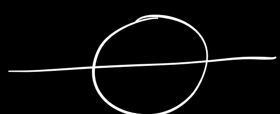
$$\frac{\lambda}{4!}\mu^{\varepsilon}(\phi_i\phi_i)^2 = \frac{\lambda_{ijkl}}{4!}\phi_i\phi_j\phi_k\phi_l$$

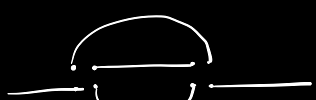
$$\lambda_{ijkl} = \frac{1}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$\mathcal{L}_c = \frac{1}{2}\delta_\phi(\partial\phi_i)^2$$

- Compute the anomalous dimension of ϕ.

The anomalous dimension is generated at two loops.



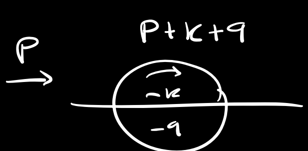


$$\left(\frac{1}{4!}\right)^2\frac{1}{2}\quad 8\cdot 4\cdot 3\cdot 2 = \frac{1}{6}$$

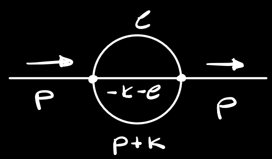
$$\frac{\lambda_{ipqr}\lambda_{pqrj}}{\lambda^2} = \frac{1}{9}(\delta_{ip}\delta_{qr} + \delta_{iq}\delta_{pr} + \delta_{ir}\delta_{pq})(\delta_{jp}\delta_{qr} + \delta_{jq}\delta_{pr} + \delta_{jr}\delta_{pq})$$

$$= \frac{1}{9}(N\delta_{ij} + \delta_{ij} + \delta_{ij})\cdot 3 =$$

$$= \frac{1}{3}(N+2)\delta_{ij}$$



$$= \mu^{2\varepsilon}\frac{N+2}{18}\lambda^2\delta_{ij}\int\frac{d^dk}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{1}{(p+k+q)^2}\frac{1}{k^2}\frac{1}{q^2}$$



$$= \mu^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d} \frac{d^d\ell}{(2\pi)^d} \frac{1}{(p+k)^2} \frac{1}{(k+\ell)^2} \frac{1}{\ell^2} =$$

$$= \mu^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 + x(1-x)k^2)^2} \frac{1}{(p+k)^2} =$$

$$= \mu^{2\varepsilon} \frac{1}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p+k)^2} \int_0^1 dx \, [x(1-x)k^2]^{\frac{d}{2}-2} =$$

$$= \mu^{2\varepsilon} \frac{1}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \frac{\Gamma\left(\frac{d}{2}-1\right)^2}{\Gamma(d-2)} \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p+k)^2 (k^2)^{2-\frac{d}{2}}}$$

$$= \mu^{2\varepsilon} \frac{1}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \frac{\Gamma\left(\frac{d}{2}-1\right)^2}{\Gamma(d-2)} \frac{\Gamma\left(3-\frac{d}{2}\right)}{\Gamma\left(2-\frac{d}{2}\right)} \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{(1-x)^{1-\frac{d}{2}}}{[k^2 + x(1-x)p^2]^{3-\frac{d}{2}}}$$

$$= \mu^{2\varepsilon} \frac{1}{(4\pi)^d} \Gamma\left(2-\frac{d}{2}\right) \frac{\Gamma\left(\frac{d}{2}-1\right)^2}{\Gamma(d-2)} \frac{\Gamma\left(3-\frac{d}{2}\right)}{\Gamma\left(2-\frac{d}{2}\right)} \frac{\Gamma(3-d)}{\Gamma\left(3-\frac{d}{2}\right)} \times$$

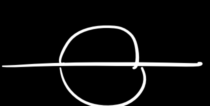
$$\times \int_0^1 dx \, (1-x)^{1-d/2} [x(1-x)p^2]^{d-3} =$$

$$= \mu^{2\varepsilon} \frac{1}{(4\pi)^d} \frac{\Gamma\left(\frac{d}{2}-1\right)^2}{\Gamma(d-2)} \Gamma(3-d) \frac{\Gamma(d-2)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{3}{2}d-3\right)} (p^2)^{d-3} =$$

$$= \frac{1}{(4\pi)^4} \frac{\Gamma\left(\frac{d}{2}-1\right)^3}{\Gamma\left(\frac{3}{2}d-3\right)} \Gamma(3-d) \, p^2 \left(\frac{p^2}{4\pi\mu^2}\right)^{d-4} =$$

$$= \frac{1}{(4\pi)^4} p^2 \left(-\frac{1}{2\varepsilon} + \frac{1}{8}(-13+6\gamma) + \frac{1}{2}\ln\left(\frac{p^2}{4\pi\mu^2}\right) + O(\varepsilon)\right) =$$

$$= \frac{1}{(4\pi)^4} p^2 \left(-\frac{1}{2}\frac{1}{\varepsilon} + \frac{1}{8}(-13+6\gamma) + \frac{1}{2}\ln\left(\frac{p^2}{4\pi\mu^2}\right) + O(\varepsilon)\right)$$

Div $\Big[$  $\Big]$ $= -\dfrac{N+2}{36}\left(\dfrac{\lambda}{16\pi^2}\right)^2 \dfrac{1}{\varepsilon} \, p^2 \, \delta_{ij}$

 $-\delta_\phi p^2$

In MS

$$\delta_\phi = -\frac{N+2}{36}\left(\frac{\lambda}{16\pi^2}\right)^2\frac{1}{\varepsilon}$$

$$Z_\phi = 1 - \frac{N+2}{36}\left(\frac{\lambda}{16\pi^2}\right)^2\frac{1}{\varepsilon} + \cdots$$

$$\beta_\lambda(\lambda) = -\varepsilon\lambda + \cdots$$

$$\gamma_\phi(\lambda) = \frac{1}{2}\frac{\partial \ln Z_\phi}{\partial \ln\mu}$$

$$\ln Z_\phi = -\frac{N+2}{36}\left(\frac{\lambda}{16\pi^2}\right)^2\frac{1}{\varepsilon} + \cdots$$

$$\mu\frac{\partial}{\partial\mu}\ln Z_\phi = -\frac{N+2}{36}\frac{2\lambda}{(16\pi^2)^2}\frac{1}{\varepsilon}\beta_\lambda + \cdots =$$

$$= \frac{N+2}{18}\left(\frac{\lambda}{16\pi^2}\right)^2 + \cdots$$

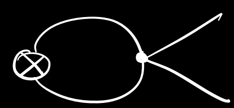
$$\gamma_\phi(\lambda) = \frac{N+2}{36}\left(\frac{\lambda}{16\pi^2}\right)^2 + \cdots$$

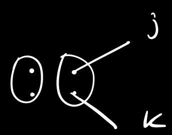
- Compute the anomalous dimension of ϕ^2

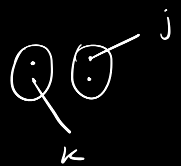
Let's consider the correlator $\qquad \phi^2 = \sum_i \phi_i^2$

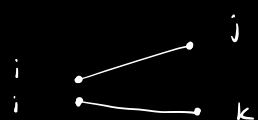
$$\langle \phi^2 \phi_j \phi_k \rangle$$

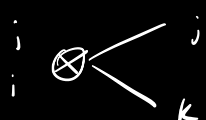
The one loop contribution is



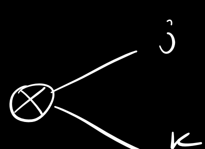
:  $\qquad \frac{1}{4!}\, 4 \cdot \begin{cases} \delta_{jk} \cdot 2\, \delta_{ii} \\ 2 \cdot 2\, \delta_{ij}\, \delta_{jk} \end{cases} = \frac{1}{3}\left(\delta_{jk}\delta_{ii} + 2\delta_{ij}\delta_{jk}\right)$

: 

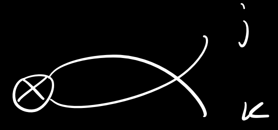
 $= 2\, \delta_{ij}\, \delta_{ik}\, \frac{1}{P_1^2}\, \frac{1}{P_2^2}$



Tree level

 $= 2\, \delta_{jk}\, \frac{1}{P_1^2}\, \frac{1}{P_2^2}$

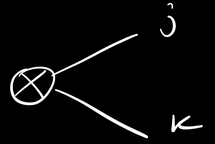
1 loop



$$= \mu^{\varepsilon} \frac{(-\lambda)}{3} \frac{(N+2)}{p_1^2\, p_2^2} \delta_{jk} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(p+q)^2} \frac{1}{q^2}$$

$$= -\frac{\lambda}{3} \frac{(N+2)}{16\pi^2} \delta_{jk} \left[\frac{2}{\varepsilon} + \text{finite terms} \right] \frac{1}{p_1^2} \frac{1}{p_2^2}$$

Counterterm

δ_ϕ 

$$= 2\, \delta_{jk} \frac{1}{p_1^2} \frac{1}{p_2^2} \delta_{\phi^2}$$

$$\delta_{\phi^2} = -\frac{N+2}{3} \frac{\lambda}{16\pi^2} \frac{1}{\varepsilon}$$

$$Z_{\phi^2} \langle [\phi^2] \ldots \rangle = \langle \phi^2 \ldots \rangle$$

$$\gamma_{\phi^2} = \mu \frac{d}{d\mu} \ln Z_{\phi^2} = -\frac{N+2}{3} \frac{1}{\varepsilon} \frac{\beta_\lambda}{16\pi^2} + \ldots$$

$$= \frac{N+2}{3} \frac{\lambda}{16\pi^2} + \ldots$$