HW 2

Compute explicitly the wave functions in the helicity basis for the following particles:

- A massive spin 1, interpolated by a vector field $A^\mu(x)$

$$\varepsilon^\mu(\vec{p},\lambda) = \langle 0|A^\mu(0)|\vec{p},\lambda\rangle, \tag{1}$$

with normalisation $\varepsilon^\dagger_\mu \varepsilon^\mu = -1$.

$$\langle 0|\, A^\mu(x)\, |p\,\lambda\rangle = \underset{\substack{\uparrow \\ \text{Dynamics}}}{\mathcal{Z}^{1/2}(m)}\; \underset{\substack{\uparrow \\ \text{Kinematics}}}{\mathcal{E}^\mu(p,\lambda)}\; e^{-ip\cdot x}$$

Wave function

$$\langle 0|\, A^\mu(0)\, |p\lambda\rangle = \langle 0|\, A^\mu(0)\, U(\Lambda_p)\, |\bar{p}\lambda\rangle$$

$$= \langle 0|\, U^\dagger(\Lambda_p)\, A^\mu(0)\, U(\Lambda_p)\, |\bar{p}\,\lambda\rangle$$

Invariance of $|0\rangle$

$$= {\Lambda_p}^\mu{}_\nu\, \langle 0|\, A^\nu(0)\, |\bar{p}\,\lambda\rangle$$

Covariance of A^μ

$$\mathcal{E}^\mu(p,\lambda) = {\Lambda_p}^\mu{}_\nu\, \mathcal{E}^\nu(\bar{p}\lambda)$$

$$\bar{p}^\mu = (m,0,0,0)$$

Helicity basis

$$U(\Lambda_p) = U(R(\theta,\varphi))\, U(B(\eta,\hat{n}_3))$$

$$R(\theta,\varphi) = e^{-i\varphi J_3}\, e^{-i\theta J_2}\, e^{i\varphi J_3}$$

$$\Lambda_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\theta c_\phi^2 + s_\phi^2 & -c_\phi s_\phi + c_\theta c_\phi s_\phi & c_\phi s_\theta \\ 0 & -c_\phi s_\phi + c_\theta s_\phi c_\phi & c_\phi^2 + c_\theta s_\phi^2 & s_\theta s_\phi \\ 0 & -c_\phi s_\theta & -s_\theta s_\phi & c_\theta \end{pmatrix}$$

$$\begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}$$

We only need $\varepsilon^\mu(\bar{p},\lambda) \leftarrow$ Basis for a $J=1$ rep of $SO(3)$

$$\langle 0| A^\mu(0)\, J^3 |\bar{p}\lambda\rangle = \langle 0|A^\mu(0)|\bar{p}\lambda\rangle\, \lambda$$

$$= \langle 0|[A^\mu(0), J^3]|\bar{p}\lambda\rangle$$

$$= (J^3)^\mu{}_\nu \langle 0|A^\nu(0)|\bar{p}\lambda\rangle$$

$$(J^3)^\mu{}_\nu\, \varepsilon^\nu(\bar{p},\lambda) = \lambda\, \varepsilon^\mu(\bar{p},\lambda)$$

$$(J^\pm)^\mu{}_\nu\, \varepsilon^\nu(\bar{p},\lambda) = \sqrt{(1\mp\lambda)(1\pm\lambda+1)}\;\; \varepsilon^\mu(\bar{p},\lambda\pm 1)$$

Asking for the normalization $\varepsilon^*(\bar{p},\lambda)\cdot\varepsilon(\bar{p},\lambda) = 1$
we find (up to a phase)

$$\varepsilon^\mu(\bar{p},+1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix} \quad \varepsilon^\mu(\bar{p},0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \varepsilon^\mu(\bar{p},0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix}$$

Applying the Helicity basis transformation we find

$$\varepsilon^\mu_+ = -(\varepsilon^\mu_-)^* = \frac{e^{i\phi}}{\sqrt{2}}\begin{bmatrix} 0 \\ -\cos\theta\cos\phi + i\sin\phi \\ -i\cos\phi - \cos\theta\sin\phi \\ \sin\theta \end{bmatrix}, \qquad \varepsilon^\mu_0 = \frac{1}{m}\begin{bmatrix} p \\ \frac{E}{p}\vec{p} \end{bmatrix}$$

- A massive spin 1/2 particle and its anti-particle, mediated by the Dirac (4-component) spinor

$$u(\vec{p},\lambda) = \langle 0|\Psi(0)|\vec{p},\lambda;r\rangle, \quad \text{and} \quad v^*(\vec{p},\lambda) = \langle 0|\Psi^*(0)|\vec{p},\lambda;\bar{r}\rangle, \tag{2}$$

with normalization $u^\dagger u = v^\dagger v = 2E$, where states with r and $\bar{r}$ (corresponding to particle and antiparticle) are related via charge conjugation

$$\mathcal{C}|\vec{p},\lambda;r\rangle = |\vec{p},\lambda;\bar{r}\rangle \tag{3}$$

whose action on the Dirac spinor is given by

$$\Psi^c(x) \equiv U(\mathcal{C}^\dagger)\Psi(x)U(\mathcal{C}) = -i\gamma^2\Psi^*(x). \tag{4}$$

You may want to also use parity transformation for particles at rest

$$\mathcal{P}|\bar{p},\lambda;r\rangle = |\bar{p},\lambda;r\rangle, \tag{5}$$

and for the Dirac field

$$U(\mathcal{P}^\dagger)\Psi(t,\vec{x})U(\mathcal{P}) = \gamma^0\Psi(t,-\vec{x}). \tag{6}$$

$$\langle 0|\psi(0)|P\lambda;r\rangle = Z^{1/2}(m)\, u(P,\lambda)$$

$$\langle 0|\psi^*(0)|P\lambda;\bar{r}\rangle = Z^{1/2}(m)\, v^*(P,\lambda)$$

Charge conjugation

$$U(C)|P\lambda r\rangle = |P\lambda\bar{r}\rangle$$

$$U^+(C)\psi(0)U(C) = -i\gamma^2\psi^*(0)$$

Relation between u and v

$$\langle 0|\psi^*(0)|P\lambda\bar{r}\rangle = \langle 0|\psi^*(0)U(C)|P\lambda r\rangle$$

$$= \langle 0|U^+(C)\psi^*(0)U(C)|P\lambda r\rangle$$

$$= -i\gamma^2\langle 0|\psi(0)|P\lambda r\rangle =$$

$$= -i\gamma^2 u^*(P,\lambda)$$

We only need to compute u.

$$u \sim \left(\tfrac{1}{2},0\right) \oplus \left(0,\tfrac{1}{2}\right)$$

Reducible rep. of Lorentz $SO(1,3)^{\uparrow}$

$$u = \begin{pmatrix} u_L \\ u_R \end{pmatrix} \qquad u(P,\lambda) = D(\Lambda_P)\, u(\bar{P},\lambda)$$

$$D(\Lambda) = D(R)\, D(B)$$

$$D(R) = \begin{pmatrix} c\theta/2 & -e^{-i\phi} s\frac{\theta}{2} \\ e^{i\phi} s\frac{\theta}{2} & c\frac{\theta}{2} \end{pmatrix} \otimes \mathbb{1}_{2\times2}$$

$$D(B) = \begin{pmatrix} e^{-\eta\sigma_3/2} & 0 \\ 0 & e^{\eta\sigma_3/2} \end{pmatrix}$$

It becomes irreducible when we consider parity

$$U^{+}(P)\,\psi(0)\,U(P) = \gamma^0 \psi(0)$$

$$U(P)\,|\bar{P}\lambda\rangle = |\bar{P}\lambda\rangle$$

$$\Downarrow$$

$$u(\bar{P}\lambda) = \gamma^0\, u(\bar{P}\lambda) \qquad \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$u_L(\bar{P},\lambda) = u_R(\bar{P}\lambda)$$

$\mathcal{U}_L(\bar{P},\lambda)$ boosts for $J=1/2$ rep of $SU(2)$

Normalization

$$\mathcal{U}_L(\bar{P},1/2) = \sqrt{m}\begin{pmatrix}1\\0\end{pmatrix} \leftarrow \quad \mathcal{U}^{+}(\bar{P},\lambda)\,\mathcal{U}^{+}(\bar{P},\lambda) = 2m$$

$$\mathcal{U}_L(\bar{P},-1/2) = \sqrt{m}\begin{pmatrix}0\\1\end{pmatrix}$$

$$\mathcal{U}(\bar{P},\lambda) = \begin{pmatrix}\sqrt{m}\,\chi_{2\lambda}(\hat{z})\\ \sqrt{m}\,\chi_{2\lambda}(\hat{z})\end{pmatrix}$$

$$\omega_{\pm} = \sqrt{E \pm |\vec{P}|} \qquad \chi_{+}(\hat{n}) = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}$$

$$\chi_{-}(\hat{n}) = \begin{pmatrix}-e^{-i\phi}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}$$

$$u_\lambda = \begin{bmatrix}\omega_{-2\lambda}\chi_{2\lambda}\\ \omega_{2\lambda}\chi_{2\lambda}\end{bmatrix}, \qquad v_\lambda = \begin{bmatrix}2\lambda\omega_{2\lambda}\chi_{-2\lambda}\\ -2\lambda\omega_{-2\lambda}\chi_{-2\lambda}\end{bmatrix}$$