HW 1A

The states of a massive particle
Spin basis

HW1

Wigner rotations for massive particles: spin basis

Compute, for states in the spin representation, the Wigner rotation matrix associated to an infinitesimal boost and verify the result given in class.

Method of induced rep.

Find a basis for states in a particular frame

$$\{ |\bar{p}, \sigma\rangle \}_{\sigma = 1, \dots, J} \qquad \text{massive}$$

$$\{ |\bar{p}, h\rangle \} \qquad \text{massless}$$

Define a transformation

$$\Lambda_p : \Lambda_p \bar{p} = p$$

and the basis of states

$$|p\sigma\rangle = U(\Lambda_p) |\bar{p}\sigma\rangle$$

States transformation under Lorentz

$$U(\Lambda)|p\sigma\rangle = U(\Lambda\Lambda_p)|\bar{p}\sigma\rangle =$$

$$= U(\Lambda_{\Lambda p}) U(\Lambda_{\Lambda p}^{-1} \Lambda \Lambda_p) |\bar{p}\sigma\rangle$$

$$= U(\Lambda_{\Lambda p}) U(W(\Lambda, p)) |\bar{p}\sigma\rangle$$

Leaves $\bar{p}$ invariant

$$W(\Lambda, p) \in \text{Little Group}(\bar{p}) = \begin{cases} SO(2) & m^2 = 0 \\ SO(3) & m^2 = 0 \end{cases}$$

$$U(\Lambda)|p\sigma\rangle = D_{\sigma'\sigma}(W(\Lambda,p))\,|\Lambda p\sigma'\rangle$$

▨ Spin basis

$$\bar{p}^\mu = (m,0,0,0)$$

$$J^3|\bar{p}\sigma\rangle = \sigma|\bar{p}\sigma\rangle$$

$$\Lambda^s_p = \exp(i\vec{\eta}\cdot\vec{K}) \qquad \hat{\eta} = \hat{p}$$

$$\eta = \cosh^{-1}\left(\frac{p^0}{m}\right)$$

- Wigner rotation for $\Lambda = R \in SO(3)$

Intended in the spin basis

$$W(R,p) = \Lambda^{-1}_{Rp}\, R\, \Lambda_p =$$

$$= \Lambda^{-1}_{Rp}\, \Lambda_{Rp}\, R = R$$

- Wigner rotation for infinitesimal boosts

$$W(\Lambda,p) = \Lambda^{-1}_{\Lambda p}\, \Lambda\, \Lambda_p$$

$$\Lambda = 1 + i\vec{\alpha}\cdot\vec{K} + O(\alpha^2)$$

$$W(\lambda,p) = \Lambda^{-1}_{\lambda p}\Lambda_p + i\alpha^i \Lambda^{-1}_p K^i \Lambda_p + \mathcal{O}(\alpha^2)$$

Two ways to proceed

I. Perform the computation by chosing a basis

II. Use the algebra and have fun with commutators

We will follow I.

$$W(\lambda,p) = \underset{\text{I}}{\Lambda^{-1}_{\lambda p}\Lambda_p} + i\alpha^i \underset{\text{II}}{\Lambda^{-1}_p K^i \Lambda_p} + \mathcal{O}(\alpha^2)$$

Let's start with the second term I

$$\Lambda_p = e^{i\vec{\eta}\cdot\vec{K}} = R\begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix} R^{-1}$$

$$= R\begin{pmatrix} \cosh\eta & \hat{n}_3 \sinh\eta \\ \hat{n}_3 \sinh\eta & \mathbb{1} + (\cosh\eta - 1)\,\hat{n}_3\hat{n}_3^T \end{pmatrix} R^{-1}$$

$R\hat{n}_3 = \hat{p}$

$$= \begin{pmatrix} \cosh\eta & \sinh\eta\,\hat{p} \\ \sinh\eta\,\hat{p} & \mathbb{1} + (\cosh\eta - 1)\,\hat{p}\hat{p}^T \end{pmatrix}$$

$$
= \begin{pmatrix} p^0/m & \frac{1}{m}\sqrt{{p^0}^2 - m^2}\ \hat{p}^T \\ \frac{1}{m}\sqrt{{p^0}^2 - m^2}\ \hat{p} & \mathbb{1} + \left(\frac{p^0}{m} - 1\right) \hat{p}\,\hat{p}^T \end{pmatrix}
$$

$$
= \begin{pmatrix} p^0/m & \frac{\vec{p}^{\,T}}{m} \\ \frac{\vec{p}}{m} & \mathbb{1} + \frac{p^0 - m}{m} \frac{\vec{p}\,\vec{p}^{\,T}}{{p^0}^2 - m^2} \end{pmatrix}
$$

$$
\Lambda_p = \begin{pmatrix} \frac{p^0}{m} & \frac{\vec{p}^{\,T}}{m} \\ \frac{\vec{p}}{m} & \mathbb{1} + \frac{\vec{p}\,\vec{p}^{\,T}}{m(p^0 + m)} \end{pmatrix} \qquad (*)
$$

Moreover we find

$$
\Lambda p = (1 + i\alpha^i K^i + O(\alpha^2))\, p
$$

$$
= \left(\mathbb{1} + \begin{pmatrix} 0 & \vec{\alpha}^{\,T} \\ \vec{\alpha} & 0 \end{pmatrix} + O(\alpha^2) \right) \begin{pmatrix} p^0 \\ \vec{p} \end{pmatrix}
$$

$$
= \begin{pmatrix} p^0 + \vec{\alpha}\cdot\vec{p} \\ \vec{p} + \vec{\alpha}\, p^0 \end{pmatrix} + O(\alpha^2)
$$

$$
i K_{\hat{n}} = \begin{pmatrix} 0 & \hat{n}^T \\ \hat{n} & 0_{3\times 3} \end{pmatrix}
$$

$$\Lambda^{-1}_{\lambda p} = \begin{pmatrix} \frac{(\lambda p)^0}{m} & -\frac{(\overrightarrow{\lambda p})^T}{m} \\ -\frac{(\overrightarrow{\lambda p})}{m} & \mathbb{1} + \frac{(\overrightarrow{\lambda p})(\overrightarrow{\lambda p})^T}{m(\lambda p^0 + m)} \end{pmatrix}$$

$$= \Lambda^{-1}_{p} + \begin{pmatrix} \frac{\vec{p}}{m}\cdot\vec{\alpha} & -\frac{p^0}{m}\vec{\alpha}^T \\ -\frac{p^0}{m}\vec{\alpha} & \frac{p^0}{m}\frac{\vec{p}\vec{\alpha}^T + \vec{\alpha}\vec{p}^T}{p^0+m} - \frac{(\vec{p}\cdot\vec{\alpha})\vec{p}\vec{p}^T}{m(p^0+m)} \end{pmatrix} + \mathcal{O}(\alpha^2)$$

$$\Lambda^{-1}_{\lambda p}\Lambda_p = \mathbb{1} +$$

$$\begin{pmatrix} \frac{\vec{p}}{m}\cdot\vec{\alpha} & -\frac{p^0}{m}\vec{\alpha}^T \\ -\frac{p^0}{m}\vec{\alpha} & \frac{p^0}{m}\frac{\vec{p}\vec{\alpha}^T + \vec{\alpha}\vec{p}^T}{p^0+m} - \frac{(\vec{p}\cdot\vec{\alpha})\vec{p}\vec{p}^T}{m(p^0+m)} \end{pmatrix}$$

$$\begin{pmatrix} \frac{p^0}{m} & \frac{\vec{p}^T}{m} \\ \frac{\vec{p}}{m} & \mathbb{1} + \frac{\vec{p}\vec{p}^T}{m(p^0+m)} \end{pmatrix} + \mathcal{O}(\alpha^2)$$

$$\Lambda^{-1}_{\lambda p}\Lambda_p = \mathbb{1} + \begin{pmatrix} 0 & \frac{\vec{p}\cdot\vec{\alpha}}{p^0+m}\vec{p}^T - \frac{p^0}{m}\vec{\alpha} \\ '' & \frac{p^0}{m}\,\frac{\vec{p}\vec{\alpha}^T - \vec{\alpha}\vec{p}^T}{p^0+m} \end{pmatrix}$$

Second part (II)

$$\Lambda^{-1}_p (i\vec{\alpha}\cdot\vec{K})\Lambda_p = \begin{pmatrix} 0 & \frac{p^0}{m} - \frac{\vec{p}\cdot\vec{\alpha}}{p^0+m}\vec{p}^T \\ '' & \frac{\vec{\alpha}\vec{p}^T - \vec{p}\vec{\alpha}^T}{m} \end{pmatrix}$$

Adding up the two terms $W = (I) + (II)$

$$W = \mathbb{1} + \begin{pmatrix} 0 & \vec{0}^T \\ \vec{0} & \frac{\vec{\alpha}\vec{p}^T - \vec{p}\vec{\alpha}^T}{p^0+m} \end{pmatrix} + O(\alpha^2)$$

$$\boxed{W = \mathbb{1} + i\,\frac{(\alpha\wedge\vec{p})}{p^0+m}\cdot\vec{J} + O(\alpha^2)}$$

Valid in all basis

HW 1B

Helicity basis vs Spin basis

HW2

Wigner rotations for massive particles: helicity basis

Using the relation between the spin and the helicity representation of the states, derive the Wigner rotation associated to an infinitesimal rotation in the helicity representation and, using the result of HW1, the one associated to an infinitesimal boost. Verify explicitly that rotations preserve the helicity, while boosts do not, in general. Check that instead if the particle is highly energetic, $E \gg m$, the helicity is preserved also by boosts.

Helicity basis

$$J^3 |\vec{p}\,\sigma\rangle = \sigma |\vec{p}\,\sigma\rangle$$

$$\Lambda_p = R(\theta,\varphi) B(\eta, \hat{n}_3)$$

$$\eta = \cosh^{-1}\left(\frac{p^0}{m}\right) \qquad \vec{p} = |\vec{p}| (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$$

$$R(\theta,\varphi) = e^{-i\varphi J^3} e^{-i\theta J^2} e^{i\varphi J^3}$$

Relation between spin and helicity basis

$$\Lambda^H_p = R(\theta,\varphi) B(\eta, \hat{n}_3) =$$

$$= B(\eta, \hat{p}) R(\theta,\varphi) = \Lambda^S_p R(\theta,\varphi)$$

$$\boxed{\Lambda^H_p = \Lambda^S_p R_{SH}(p)}$$

Relation between Wigner rotations

$$W^H = (\Lambda^H_{\lambda p})^{-1} \Lambda \Lambda^H_p =$$

$$= R^{-1}_{SH}(\Lambda p)(\Lambda^S_{\Lambda p})^{-1} \Lambda \Lambda^S_p R_{SH}(p)$$

$$= R^{-1}_{SH}(\Lambda p) W^S R_{SH}(p)$$

$$\boxed{W^H = R^{-1}_{SH}(\Lambda p) W^S R_{SH}(p)}$$

- Wigner rotation for infinitesimal rotations

$\Lambda \in SO(3)$ $\quad\quad \Lambda = 1 - i \vec{\alpha} \cdot \vec{J} + O(\alpha^2)$

We need to compute

① $R^{-1}_{SH}(\Lambda p)$ $\quad\quad$ ⑪ W^S

We have seen in the previous exercise that

$$W^S = \Lambda = 1 - i \alpha^i J^i + O(\alpha^2)$$

$$= \begin{pmatrix} 1 & & \vec{0}^T & \\ & 1 & -\alpha^3 & \alpha^2 \\ \vec{0} & \alpha^3 & 1 & -\alpha^1 \\ & -\alpha^2 & \alpha^1 & 1 \end{pmatrix} + O(\alpha^2)$$

$p \to \Lambda p$

$(\theta, \varphi) \to (\theta', \varphi')$

$$\begin{pmatrix} \cos\varphi' \sin\theta' \\ \sin\varphi' \sin\theta' \\ \cos\theta' \end{pmatrix} = (1 - i \vec{\alpha} \cdot \vec{J}) \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix} + O(\alpha^2)$$

$\theta' = \theta + \delta\theta$

$\varphi' = \varphi + \delta\varphi$

- Expand in $\delta\theta, \delta\varphi$ and match to leading order in α.

$$\delta\theta = \alpha^2 \cos\varphi - \alpha^1 \sin\varphi$$
$$\delta\varphi = \alpha^3 - \operatorname{cotan}\theta\,(\alpha^1 \cos\varphi + \alpha^2 \sin\varphi)$$

$$R^{-1}_{SH}(\Lambda p) = e^{-i(\varphi+\delta\varphi)J^3}\, e^{i(\theta+\delta\theta)J^2}\, e^{i(\varphi+\delta\varphi)J^3}$$

$$= R^{-1}_{SH}(p) - i\,\delta\varphi\, J^3 R^{-1}_{SH}(p)$$
$$+ i\,\delta\theta\, e^{-i\varphi J^3} e^{i\theta J^2} J^2 e^{i\varphi J^3}$$
$$+ i\,\delta\varphi\, R^{-1}_{SH}(p)\, J^3 + \dots$$

$$W_H = R^{-1}_{SH}(\Lambda p)\,(1 - i\vec{\alpha}\cdot\vec{J})\, R_{SH}(p) + O(\alpha^2)$$

$$= \mathbb{1} - i\, J^3\left(\alpha^3 + \alpha^1 \frac{\hat{p}^1}{1+\hat{p}^3} + \frac{\alpha^2\hat{p}^2}{1+\hat{p}^3}\right) + O(\alpha^2)$$

$$= \mathbb{1} - i\, J^3\left(\alpha^3 + \frac{\vec{\alpha}\cdot\hat{p}}{1+\hat{p}^3}\right) + O(\alpha^2)$$

The Wigner rotation associated to a rotation does not mix the helicities.

- Wigner rotation for infinitesimal boosts

$$\Lambda = \mathbb{1} + i\vec{\alpha}\cdot\vec{K} + O(\alpha^2)$$

$$W^s = \mathbb{1} + i\,\frac{\vec{\alpha}\wedge\vec{P}}{P^0+m}\cdot\vec{J} + O(\alpha^2)$$

$$W^H = R^{-1}_{SH}(\Lambda P)\,W^s\,R_{SH}(P)$$

$$\delta\theta = -\frac{P^0}{|\vec{P}|}\left(\alpha^3\sin\theta - \cos\theta\,(\alpha^1\cos\varphi + \alpha^2\sin\varphi)\right)$$

$$\delta\varphi = \frac{P^0}{|\vec{P}|}\frac{1}{\sin\theta}\left(\cos\varphi\,\alpha^2 - \sin\varphi\,\alpha^1\right)$$

In the high energy limit

$$W_H = 1 - i\,\frac{(\alpha\wedge\hat{P})^3}{1+\hat{P}^3}\,J^3 + O(\alpha^2)$$