EPFL, QFT-III, January 2026

AQFT Written Exam

The written part of the exam consists of two problems. Provide detailed written solutions and return them by the deadline.

Those marked by $*$ are bonus questions: if you skip them you can still get the top grade by answering properly to the other questions, while a wrong answer to them won't be "held against you"

Problem 1: Ward identity for translations and dilatations

Consider a Poincaré invariant QFT with energy-momentum $T^{\mu\nu}$ and invariant vacuum $|0\rangle$.

a) Given a real scalar operator $\mathcal{O}(x) = \mathcal{O}^\dagger(x)$:

- Write down the spacetime translation Ward identity for the time ordered correlator $\langle 0|T\left[T^{\mu\nu}(x)\mathcal{O}(y_1)\ldots\mathcal{O}(y_n)\right]|0\rangle \equiv G_n^{\mu\nu}(x, y_1, \ldots y_n)$, i.e. express $(\partial/\partial x^\mu)G_n^{\mu\nu}$ in terms of $n-$point functions. Comment on the particular case $n = 1$.

- Using the Ward identity show that, by the addition of suitable contact terms, the Källén-Lehmann decomposition of $G^{\mu\nu}(x-y) \equiv G_1^{\mu\nu}(x, y) = \langle 0|T\left[T^{\mu\nu}(x)\mathcal{O}(y)\right]|0\rangle$ reads

  $$G^{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{ipx} \int dm^2\, \rho(m^2)\frac{\eta_{\mu\nu}p^2 - p^\mu p^\nu}{p^2 - m^2 + i\epsilon} = \int \frac{d^4p}{(2\pi)^4}\, e^{ipx} a(p^2)(\eta_{\mu\nu}p^2 - p^\mu p^\nu) \tag{1}$$

  Remark: As usual the 2-point function constructed via the KL decomposition, which uses the operator description, differs by contact terms from the proper one obtained through the path integral.

  Hint: you may want to use CPT invariance: there is an anti-unitary operator $U_{\rm CPT}$ such that

  $$U_{\rm CPT}^{-1}\mathcal{O}(0)U_{\rm CPT} = \mathcal{O}(0)\,, \quad U_{\rm CPT}^{-1}T^{\mu\nu}(0)U_{\rm CPT} = T^{\mu\nu}(0) \tag{2}$$

  $$U_{\rm CPT}|\mathbf{k}, j, \sigma; q\rangle = |\mathbf{k}, j-\sigma; \bar{q}\rangle(-1)^\sigma \times \eta\,, \quad U_{\rm CPT}|\mathbf{k}, \lambda; q\rangle = |\mathbf{k}, -\lambda; \bar{q}\rangle(-1)^\lambda \times \eta \tag{3}$$

  where the last line refers to respectively massive and massless states (the η's are σ- and λ-independent conventional phases).

- * Assuming P and T are also symmetries, what can you say about the quantum numbers of the intermediate states in the KL decomposition?

b) Assume now that the given QFT is also invariant under dilatations, with the associated conserved current S^μ. Given a scalar field $\mathcal{O}$ transforming under dilatation as $\mathcal{O}(x) \to \lambda^\Delta \mathcal{O}(\lambda x)$ with $\Delta \neq 0$, write the dilatation Ward identity for $\langle 0|T\left[S^\mu(x)\mathcal{O}(y)\right]|0\rangle$.

c) Consider now the more specific case where $S^\mu = x_\nu T^{\mu\nu}$. What does the dilatation Ward identity imply for the spectral density $\rho(m^2)$ in the following two cases?

1. $\langle 0|\mathcal{O}|0\rangle = 0$.
2. $\langle 0|\mathcal{O}|0\rangle = M^\Delta \neq 0$.

Comment on the result.

d*) Make all the above concrete, with explicit expressions for $T^{\mu\nu}$ and $\rho(m^2)$, in the two cases of massive and massless free real scalar ϕ in arbitrary D-dimensional spacetime with $\mathcal{O} = \phi^2$.
Hint: in the massless case you must of course consider the improved traceless $T^{\mu\nu}$.

Problem 2: Renormalization in a specimen of Yukawa theory

Consider a 4D QFT of one complex scalar field ϕ and N Dirac fermions ψ_a ($a = 1, \dots, N$) with Lagrangian (here on, when we write a Lagrangian, we mean the renormalized Lagrangian for some choice of the renormalization scale μ)

$$\mathcal{L} = \partial\phi^*\partial\phi + \bar{\psi}_a(i\partial\!\!\!/ - m)\psi_a - M^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2 + \big(y\phi\,\bar{\psi}_a U_{ab}\psi_b + \text{h.c.}\big) \tag{4}$$

where the sum over repeated indices is understood. The matrix U_{ab} is given by diag$(1, \omega, \omega^2,$-$\dots, \omega^{N-1})$ with $\omega = e^{2\pi i/N}$ (equivalently: $U_{ab} = \delta_{ab}\omega^{a-1}$, with no summation). Notice that $U^N = \mathbf{1}$.

a) Discuss the global symmetries for each choice of N.
(*Remark: the symmetry group, while being non-abelian, is nonetheless the semidirect product of two abelian factors. To deduce the consequences of symmetry it is enough to consider such abelian factors.*)

b) Study renormalization for $N = 3$ and $N = 4$:

- According to symmetry and power counting, which counterterms are needed? Equivalently: what is the complete set of couplings that is needed to properly renormalize the theory?
- Considering such complete set, compute the full set of 1-loop RGE's.

c) Discuss qualitatively what changes for $N > 4$.

d) Discuss qualitatively renormalization in the case $N = 3$ in 3D rather than 4D.

e*) Consider finally the case where you have two sets of fermions, χ_k with $k = 1, 2, 3$ and ψ_a with $a = 1, 2, 3, 4$, and where the renormalized Lagrangian at some scale μ is given by

$$\begin{aligned}\mathcal{L} &= \partial\phi^*\partial\phi + \bar{\chi}_k(i\partial\!\!\!/ - m_\psi)\chi_k + \bar{\psi}_a(i\partial\!\!\!/ - m_\psi)\psi_a - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2 \\ &+ \big[\big(y_3\phi\,\bar{\chi}_k V_{k\ell}\chi_\ell + y_4\phi\,\bar{\psi}_a U_{ab}\psi_b\big) + \text{h.c.}\big]\end{aligned} \tag{5}$$

where $V = \text{diag}(1, e^{2\pi i/3}, e^{4\pi i/3})$ while $U = \text{diag}(1, i, -1, -i)$ are just respectively the $N = 3$ and $N = 4$ instances of the matrices encountered before. Discuss symmetries and (qualitatively) renormalization. For instance: in order to encounter all allowed types of counterterms quadratic in ϕ, up to what loop order must you go?