Non-equilibrium dynamics

Exercise Sheet 10

Free fermionic models like the transverse-field Ising (TFI) model are examples of integrable models. These models have enough conserved quantities such that one can compute any observable $O(t)$ analytically or numerically in polynomial time even for a large number of degrees of freedom. In contrast, strongly-interacting physical systems are (mostly) non-integrable, and the problem of solving the system is exponentially hard on the number of degrees of freedom.

Integrable and non-integrable quantum many-body systems are also differentiated by the distributions of the level spacings $\delta\varepsilon$ of their eigenspectrums. The conserved quantities in integrable models allow for level crossings and less level repulsion in general, while their absence in non-integrable models means most levels repel each other and thus there is an absence of level spacings close to zero.

In the following exercises we will numerically compare the energy level spacings of integrable and non-integrable spin models, and random matrices.

1. Consider an Ising ring with site-dependent longitudinal and transverse field components:

$$\hat{\mathcal{H}} = -\sum_{l=1}^{L} \hat{\sigma}_l^z \hat{\sigma}_{l+1}^z + h_x(l)\hat{\sigma}_l^x + h_z(l)\hat{\sigma}_l^z, \tag{1}$$

where

$$h_x(l) = h_x + X(l), \quad h_z(l) = h_z + Z(l), \tag{2}$$

where X and Z are normal-distributed random variables of standard deviations σ_X and σ_Y which are small compared to the uniform couplings. We added disorder on the field couplings to remove symmetries like reflections and translations from the model.

(a) Taking into account the Jordan-Wigner transformation to spinless fermions, in what limit(s) is the Hamiltonian (1) quadratic in fermion operators (disregarding the boundary conditions)?

(b) Write a code that will produce the Hamiltonian as a $2^L \times 2^L$ matrix in the $\hat{\sigma}^z$ configuration basis.

Suggestion: The bit representation of each integer from 0 to $2^L - 1$ corresponds can be seen as a spin configuration. The $\hat{\sigma}^z$ terms are diagonal in this basis, while the $\hat{\sigma}^x$ terms flip single "bits", giving off-diagonal terms. The provided script ex.py defines aptly named bit manipulation functions that may be useful in completing the Hamiltonian matrix.

(c) Prepare the code to diagonalize the Hamiltonian and to calculate and plot the distribution of energy spacings $\delta\varepsilon$ for some set of parameters.

Suggestion: Generate $\delta\varepsilon$ for several realizations of X and Z and generate the histogram of the full set. Remove part of the lowest and largest eigenvalues of the spectrum as the gaps in the extremes tend to be much larger than in the bulk and we are mainly interested in the behavior close to $\delta\varepsilon \sim 0$.

(d) Investigate the form of the $\delta\varepsilon$ distribution in the free fermion limit determined in point (a), with and without disorder in the couplings.

Suggestion: Take $L \approx 10$ and average over ~ 40 disorder realizations.

(e) Away from the limit above, the model is no longer integrable. Check that there is significant level repulsion in this interacting case (at least when the system is disordered).

2. The level spacing distribution of strongly interacting systems can be modeled through random matrices (RM). Consider a $2^L \times 2^L$ symmetric matrix M whose matrix elements are normal random variables of variance 1.

   (a) Calculate the $\delta\varepsilon$ distribution while averaging over many realizations of M and check that it approximately follows the following distribution suggested by Wigner:

$$p(s) \sim s e^{-s^2}, \tag{3}$$

   where s is a normalized level spacing, $s = \delta\varepsilon/x$. What should x be to match with the observed distribution?

   Suggestion: Complete the ex.py script.

   (b) Compare the distribution (3) to the distributions obtained for non-integrable models in question 1, again trying to adjust x to obtain the best match.