# Topological Order

Exercise Sheet 9

In free fermion symmetry-protected topological (SPT) systems, we can define a bulk topological invariant (BTI) which is usually a functional of the bulk dispersion or parts of it. The exact form of the BTI is dependent on the set of discrete symmetries present that protect the topological phase.

The BTI matches the edge topological invariant (ETI) that arises when we consider open boundary conditions on a semi-infinite limit. This matching is called the bulk-edge correspondence, which we assume without proof.

In this exercise we will see how the BTI of systems with time-reversal $\mathcal{T}$ and particle-hole $\mathcal{P}$ symmetry arises. These operators are anti-linear, meaning they do not commute with constants $c$ but instead conjugate them. Time-reversal acts trivially on real-space fermions $\hat{a}_l^\dagger$ while $\mathcal{P}$ exchanges $\hat{a}_l^\dagger \leftrightarrow \hat{a}_l$, so we may write

$$\mathcal{T} c\hat{a}_l^\dagger = c^* \hat{a}_l^\dagger \mathcal{T}, \quad \mathcal{P} c\hat{a}_l^\dagger = c^* \hat{a}_l \mathcal{P}, \quad \mathcal{T}^2 = \mathcal{P}^2 = 1. \tag{1}$$

The action on momentum-space fermions $\hat{a}_k^\dagger$ may not be the same but follows from equation (1). Let us take a general translation invariant free fermion Hamiltonian with a single band:

$$\hat{\mathcal{H}} = \frac{1}{2\pi}\int dk\ [x(k) - iy(k)]\hat{a}_k^\dagger \hat{a}_{-k}^\dagger + h.c. + z(k)(\hat{a}_k^\dagger \hat{a}_k - \hat{a}_{-k}\hat{a}_{-k}^\dagger) + w(k)(\hat{a}_k^\dagger \hat{a}_k + \hat{a}_{-k}\hat{a}_{-k}^\dagger) \tag{2}$$

$$= \frac{1}{2\pi}\int dk\ \begin{pmatrix}\hat{a}_k^\dagger & \hat{a}_{-k}\end{pmatrix} M(k) \begin{pmatrix}\hat{a}_k \\ \hat{a}_{-k}^\dagger\end{pmatrix}, \tag{3}$$

where $x, y, z, w$, are real so that $\hat{\mathcal{H}}$ is Hermitian, and

$$M(k) \equiv x(k)\hat{\sigma}^x + y(k)\hat{\sigma}^y + z(k)\hat{\sigma}^z + w(k)\mathbb{1}. \tag{4}$$

The eigenvalues of $M(k)$ are $\varepsilon_\pm(k) = w(k) \pm |\vec{d}(k)|$ where $\vec{d} = (x, y, z)$. From $-\pi \le k < \pi$, the vector $\vec{d}$ describes a closed line in space.

Some symmetries may force the closed line to lie on a plane. Moreover, if $w(k) = 0$, then the spectrum is gapless if and only if the closed line crosses the origin. In such conditions, one defines the BTI as the number of times the closed line winds around the origin, with loops in opposite directions having opposite signs. This BTI is known as the winding number $\mathcal{W}$. Clearly, the winding number can only change if we tune the parameters such that the closed line crosses the origin (gap closing), unless we go out of the plane (Fig. 1).

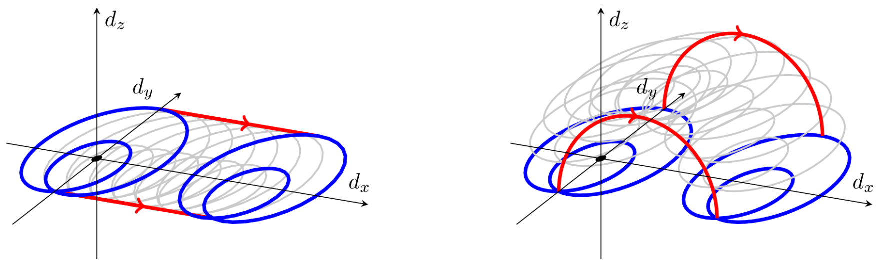


Figure 1: Winding of the vector $\vec{d}$ around the origin. Examples of deformations that preserve (left) or not (right) the in-plane symmetry.

1. Show that $x(k)$ and $y(k)$ may be taken as odd functions without loss of generality.
2. Find what conditions are imposed on $x, y, z, w$, by each of the following properties:
   (a) Time-reversal symmetry: $\mathcal{T}\hat{\mathcal{H}}\mathcal{T} = \hat{\mathcal{H}}$.
   (b) Particle-hole symmetry: $\mathcal{P}\hat{\mathcal{H}}\mathcal{P} = -\hat{\mathcal{H}}$.
   (c) Sub-lattice symmetry: $\mathcal{S}\hat{\mathcal{H}}\mathcal{S} = -\hat{\mathcal{H}}$, where $\mathcal{S}$ is linear and $\mathcal{S}c\hat{a}_l^\dagger = ce^{i\pi l}\hat{a}_l^\dagger\mathcal{S}$.
3. For what (combination of) symmetries is the winding number a BTI? Is there such a combination that guarantees $\mathcal{W} \neq 0$?
4. Take now a Kitaev chain model with next-nearest neighbor couplings:

$$\hat{\mathcal{H}} = -\sum_l \left[ \mu(\hat{a}_l^\dagger \hat{a}_l - \hat{a}_l \hat{a}_l^\dagger) + \sum_{j=1}^{2} t_j (\hat{a}_l^\dagger \hat{a}_{l+j} + \hat{a}_l^\dagger \hat{a}_{l+j}^\dagger + h.c.) \right]. \tag{5}$$

   Find $x, y, z, w$ for this model after a Fourier transform. Check that $\mathcal{W}$ is a BTI for this model.
5. Find the lines where the gap closes in the $(t_1, t_2)$ plane and calculate the winding number on each phase for
   (a) $\mu = 0$.
   (b) $\mu = 1$.

   Suggestion: Since $\mathcal{W}$ is invariant throughout the whole phase, you may calculate $\mathcal{W}$ from a trivial point or limit of the phase.
6. For phases where $|\mathcal{W}| > 1$, we predict that there will be more than one uncoupled Majorana at the edge of an open chain. How do you justify that the Majoranas are not coupled to each other given their proximity?
7. Revisit the past exercises on the SSH chain (Sheet 1, Ex. 4) and the Kitaev chain (Sheet 3, Ex. 2) and numerically investigate their topological phases. Try adding terms to the Hamiltonian that would remove the appropriate protecting symmetries.