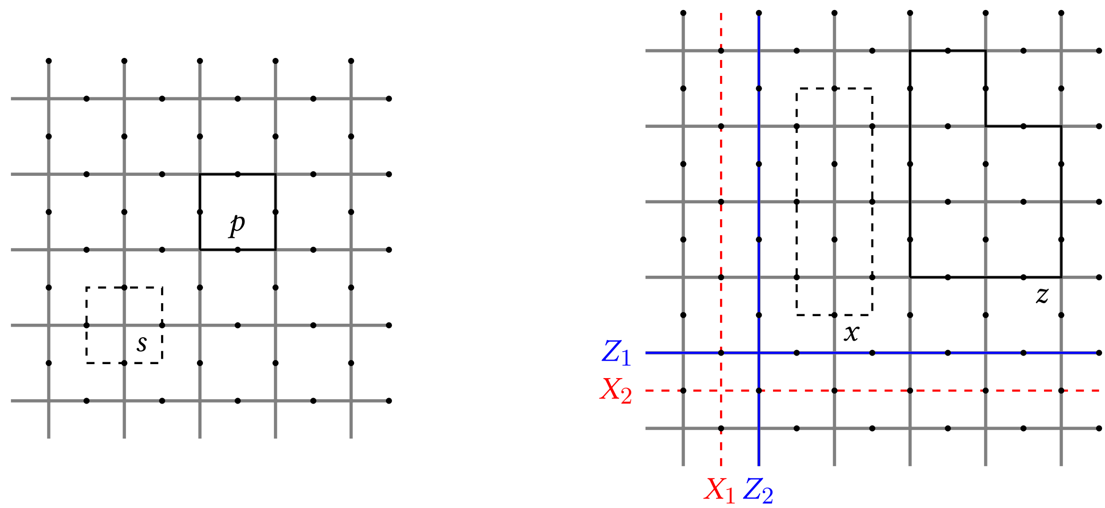
Topological Order

Exercise Sheet 8



(a) (b)

Figure 1: Toric code model on the torus. The black dots are spins. We exemplify different types of loops. To each loop we can associate an operator that is the product of the appropriate Pauli matrices ($\hat{\sigma}^x$ or $\hat{\sigma}^z$) of all spins crossed along the loop. (a) The basic plaquette and "star" loops that define the A_s and B_p operators. (b) Loop operators on the torus.

1. Consider the Toric code Hamiltonian on the torus:

$$\hat{\mathcal{H}} = -\sum_{s\in\{+\}} A_s - \sum_{p\in\{\square\}} B_p, \tag{1}$$

$$A_s \equiv \prod_{j\in s} \hat{\sigma}_j^x, \quad B_p \equiv \prod_{j\in p} \hat{\sigma}_j^z, \tag{2}$$

where $[A_s, B_p] = 0$. Products of neighboring A_s or B_p operators result in products of $\hat{\sigma}^x$ or $\hat{\sigma}^z$ along closed loops like x_l and z_l in Fig. 1b. These loop products commute with all A_s and B_p.

However, in geometries with periodic boundary conditions we can form loop products of $\hat{\sigma}^x$ or $\hat{\sigma}^z$ which are not products of A_s or B_p. These loops cross the periodic boundaries, example being the loops on the torus X_1, Z_1, X_2, Z_2 in Fig. 1b.

A ground state $|\xi\rangle$ will be eigenstate of all A_s and B_p with eigenvalue 1. Let us show that there are exactly 4 ground states and let us classify their degeneracy.

(a) By a counting of independent $A_s = B_p = 1$ conditions and degrees of freedom, show that the ground state manifold can be at most 4-fold degenerate.

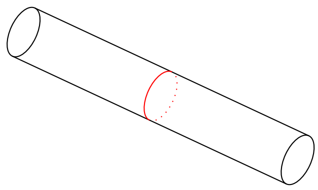
(b) Show that the X_i and Z_j operators commute with the A_s and B_p operators and thus the Hamiltonian.

(c) Given the commutators between all X_i, Z_j, divide the energy spectrum in as many sectors of $X_i, Z_j = \pm 1$ as possible.

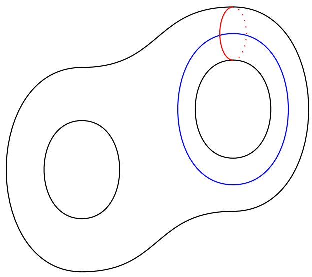
(d) Given the anti-commutators between all X_i, Z_j, show that each energy level is at least 4-fold degenerate.

(e) How can we span the whole ground state manifold starting from the ground state of one of the sectors determined in (c)?

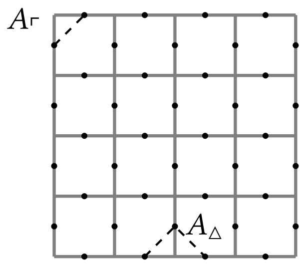
2. Determine the ground state degeneracy of the Toric code when embedded in the following geometries: a cylinder, a torus with g holes, and an open boundary square lattice. Note that at the open boundaries of the cylinder and the square lattice there are 3-spin or 2-spin A_s terms, as exemplified in (c) below.



(a) Cylinder.



(b) $g = 2$ torus.



(c) Toric code on the square lattice.

3. Consider the Toric code on the open boundary square lattice. Let us remove one or more edges (and their spins) such that we create holes inside the lattice. The affected B_p operators are also removed.

(a) Determine the ground state degeneracy for a single edge (spin) removed.

(b) We can expand a hole by removing more edges. Does the shape and size of the hole influence the ground state degeneracy?

(c) What is the ground state degeneracy if we have h holes?