Quantum phase transitions

Exercise Sheet 6

Using the Jordan-Wigner transformation between spins and spinless fermions $\hat{a}_i$, the transverse field Ising Hamiltonian becomes quadratic in the $\hat{a}_i$ and takes the form

$$\hat{\mathcal{H}} = -\sum_{i=1}^{L-1}\left(\hat{a}_i^\dagger \hat{a}_{i+1} + \hat{a}_i^\dagger \hat{a}_{i+1}^\dagger + \text{h.c.}\right) - h\sum_{i=1}^{L}\left(\hat{a}_i^\dagger \hat{a}_i - \hat{a}_i \hat{a}_i^\dagger\right). \tag{1}$$

1. This Hamiltonian does not conserve the total number of fermions due to the pair creation and annihilation terms. Show that instead the fermion parity is conserved, that is,

$$P = \prod_i^L (\hat{a}_i \hat{a}_i^\dagger - \hat{a}_i^\dagger \hat{a}_i) \tag{2}$$

commutes with the Hamiltonian (1).

2. Thanks to the fermion parity conservation and the fact that all couplings are real, the Hamiltonian can be written as

$$\hat{\mathcal{H}} = \vec{v}^\dagger M \vec{v}, \quad M \equiv \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}, \tag{3}$$

where $\vec{v}^\dagger \equiv (\hat{a}_1^\dagger, \dots, \hat{a}_L^\dagger, \hat{a}_1, \dots, \hat{a}_L) \equiv (\vec{a}^\dagger\ \vec{a})$, and where A is a symmetric matrix and B is anti-symmetric so that M is symmetric. Determine all matrix elements of A and B.

3. Given the eigenvalue equation of the matrix M which we can write as

$$\begin{pmatrix} A & B \\ -B & -A \end{pmatrix}\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \varepsilon \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, \tag{4}$$

show that the eigenvalues ε of M come in symmetric pairs $\pm\varepsilon$ and determine the relation between the eigenvectors of opposite energy.

4. We can decompose M as $U^T M U = \text{diag}(-\varepsilon_1, \dots, -\varepsilon_L, \varepsilon_1, \dots, \varepsilon_L)$, where $\varepsilon_i \geq 0$ and U is orthogonal. Writing

$$U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}, \tag{5}$$

show how the four submatrices are related given the result from question 3.

5. The transformation U brings the Hamiltonian to diagonal form

$$\hat{\mathcal{H}} = \sum_{i=1}^{L} \varepsilon_i (\hat{\mu}_i^\dagger \hat{\mu}_i - \hat{\mu}_i \hat{\mu}_i^\dagger), \qquad \begin{pmatrix} \vec{\mu} \\ \vec{\mu}^\dagger \end{pmatrix} = U^T \begin{pmatrix} \vec{a} \\ \vec{a}^\dagger \end{pmatrix}. \tag{6}$$

Show that the operators $\hat{\mu}$ obey the fermionic anti-commutation relations $\{\hat{\mu}_i, \hat{\mu}_j\} = 0$ and $\{\hat{\mu}_i^\dagger, \hat{\mu}_j\} = \delta_{ij}$.

6. Obtain the quasi-particle spectrum and the relevant sub-matrices of U numerically and study the spectrum as $h \to 1^+$. How does the energy gap scale in this region for large L?

   Suggestion: Complete the provided script ex.py.

7. What is the expected value of the ground state entanglement entropy S_l for a bipartition of the chain at a bond $l = 1, \ldots, L-1$ when the field h is close to the limits $h \to 0, \infty$?

8. At the critical point and for large enough L, the entanglement entropy S_l is known to follow the Calabrese-Cardy formula,

$$S_l = \frac{c}{6} \log\left(\frac{2L}{\pi} \sin \frac{\pi l}{L}\right) + c_1, \tag{7}$$

   where c_1 is a system-dependent constant and c is known as the *central charge*, a positive number, often a fraction, that characterizes (in this case) the 2D classical Ising universality class.

   Calculate S_l at the critical point using the correlation matrix method and compare it to Eq. (7) to determine the central charge. When constructing C_l you should consider ALL non-zero two-point correlators in the subspace of the first l sites:

$$C_l = \left\langle \begin{pmatrix} \vec{a}_l \\ \vec{a}_l^\dagger \end{pmatrix} (\vec{a}_l^\dagger \; \vec{a}_l) \right\rangle, \tag{8}$$

   where the subscript l indicates the vector runs over the operators from site 1 up to site l included, and where we take the expectation of all matrix elements on the ground state.

   Suggestion: The correlation matrix C_l can be written in terms of the submatrices of U. Try to calculate first C_L. You can complete the script ex.py to calculate the entropy.

   Note: C_l is now a $2l \times 2l$ matrix, but the entanglement spectrum of a free fermion system is uniquely determined by l numbers, equal to the number of degrees of freedom in the subsystem. This manifests in the spectrum $\{\lambda_i\}$ of C_l which is symmetric with respect to the maximum of entropy $\lambda = 1/2$ ($0 \le \lambda_i \le 1$). To avoid double counting one should either take

$$S_l = -\sum_{n=1}^{l} \lambda_n \log \lambda_n \tag{9}$$

   or, equivalently, take one of the halves of the spectrum of C_l when calculating S_l with the usual formula.