EPFL

Quantum phase transitions

Exercise Sheet 5

In this exercise we consider the transverse field Ising (TFI) model with open boundary conditions:

$$\hat{\mathcal{H}} = -\sum_{i=1}^{L-1} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - h \sum_{i=1}^{L} \hat{\sigma}_i^x, \tag{1}$$

where we take $h > 0$. There is a transformation one can apply to the TFI model which relates the ordered and disordered phases and helps us locate the quantum phase transition.

The main idea is to change to a bond picture: take new spin variables $\tilde{\sigma}^z$ that live in the bonds, that is, each bond spin is up or down whether the two regular spins of the bond are parallel or anti-parallel. This results in the relation

$$\tilde{\sigma}_i^z = \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z. \tag{2}$$

Since there is one less bond than $\hat{\sigma}_i^z$ spins, we can keep the last spin in the new basis ($\tilde{\sigma}_L^z = \hat{\sigma}_L^z$) to remove the degeneracy.

1. What is the inverse of the transformation (2)?
2. For the $\tilde{\sigma}^x$ operators, the roles are swapped: The $\hat{\sigma}^x$ are the bond spins of $\tilde{\sigma}^x$. What transformations between $\tilde{\sigma}^x$ and $\hat{\sigma}^x$ does this suggest? How can the degeneracy be resolved?
3. Choose a transformation for $\tilde{\sigma}^x$ such that the new variables are proper spin degrees of freedom, that is, $\tilde{\sigma}^x$, $\tilde{\sigma}^z$, and $\tilde{\sigma}^y \equiv i\tilde{\sigma}^x\tilde{\sigma}^z$ locally obey $\tilde{\sigma}^a\tilde{\sigma}^b = \delta_{ab} + i\varepsilon_{abc}\tilde{\sigma}^c$, while operators at different sites commute.
4. The full transformation will swap the bond and magnetic field terms of the Hamiltonian (1). What subsequent transformations are needed to bring the Hamiltonian to the original form? Can we then conclude on the location of the phase transition?