Entanglement of quantum many body systems

Exercise Sheet 4

1. Take a tight-binding model on an infinite square lattice, where the quasiparticle excitations are plane waves of momentum $\vec{k} = (k_x, k_y)$ on a square Brillouin zone between $-\pi$ and π and have energy
$$\varepsilon_{\vec{k}} = -2(\cos k_x + \cos k_y). \tag{1}$$
The two-point correlation matrix $C_{\vec{r}} \equiv \langle \hat{a}^\dagger_{\vec{r}} \hat{a}_{\vec{0}} \rangle$ in the ground state at half-filling can be shown to be
$$C_{\vec{r}} = \frac{1}{2}\frac{\sin w_+ \sin w_-}{w_+ w_-}, \qquad w_\pm \equiv \frac{\pi}{2}(x \pm y) \tag{2}$$
   (a) (Optional) Obtain Eq. (2). Note that $\varepsilon_{\vec{k}}$ is negative in the square formed by the points $(0, \pm\pi)$ and $(\pm\pi, 0)$.
   (b) Consider several subsets of sites A of variable size (connected or not) like 1D arrays on the plane, concentric squares or circles, and study the scaling of the entropy with the area and linear size of A by constructing and diagonalizing the correlation matrix.
   Suggestion: Complete the script ex3.py. Generate the list of coordinates (x, y) that belong to the considered subset of sites A and calculate $C_{\vec{r}}$ on that subset.

2. In 1987, Affleck, Kennedy, Lieb and Tasaki (AKLT) introduced a 1D model of spin-1s whose exact ground state has a simple matrix product state (MPS) representation with index rank $\chi = 2$. For a periodic ring of L spin-1s, we can write the AKLT ground state as
$$|\Psi\rangle = \sum_{\{s\}} \mathrm{Tr}(A^{s_1} A^{s_2} \cdots A^{s_L}) \, |s_1 s_2 \cdots s_L\rangle \,, \tag{3}$$
where the 2×2 matrices A^s are given by
$$A^+ = \sqrt{\frac{2}{3}}\sigma^+, \quad A^0 = -\sqrt{\frac{1}{3}}\sigma^z, \quad A^- = -\sqrt{\frac{2}{3}}\sigma^-, \tag{4}$$
where σ are Pauli matrices. We can exploit the periodic MPS form of the state to directly calculate expectation values from the A^s matrices. A useful object in such calculations is the *transfer matrix*:
$$T = \sum_s (A^s)^* \otimes A^s. \tag{5}$$
The norm of the state and other quantities like one and two-point functions reduce to chain products of T and other matrices. We have
$$\langle \Psi | \Psi \rangle = \mathrm{Tr}\left(T^L\right), \tag{6}$$
and, assuming the state is normalized,
$$\langle S^z_0 \rangle = \mathrm{Tr}\left(U T^{L-1}\right), \quad \langle S^z_0 S^z_r \rangle = \mathrm{Tr}\left(U T^{r-1} U T^{L-r-1}\right), \tag{7}$$

where

$$U = \sum_{s} (S^z)^{ss} (A^s)^* \otimes A^s. \tag{8}$$

Given the small χ of the AKLT state, it is feasible to calculate these quantities by hand.

(a) Calculate the transfer matrix T of the AKLT state and find its eigenvalues and eigenvectors.

(b) Determine the norm of the state.

(c) Calculate $\langle S_0^z \rangle$ and $\langle S_0^z S_r^z \rangle - \langle S_0^z \rangle^2$ in the $L \to \infty$ limit.