Entanglement of quantum many body systems

Exercise Sheet 3

1. Recall exercise 2 from exercise sheet 1. We have a system of N bosons on two coupled bosonic modes. This time we introduce a more natural "hopping" model and add a potential energy difference:
$$\hat{\mathcal{H}} = N^{-\frac{1}{2}}(\hat{b}_0^\dagger \hat{b}_1 + \hat{b}_1^\dagger \hat{b}_0) + V\left(\hat{b}_1^\dagger \hat{b}_1 - \hat{b}_0^\dagger \hat{b}_0\right) \tag{1}$$
We take again a bipartition between the two modes.
   (a) What is the equivalent tight-binding model this time?
   (b) How will the ground state wavefunction behave as we increase V? To what value does S tend to when $V \to \infty$?
   (c) Numerically obtain the quasi-particle solution of this system and study the scaling with N of the ground state entanglement entropy, comparing the cases $V = 0$ and $V \neq 0$.
   Suggestion: Complete the provided python script ex1.py. Take $V \in [0, 1]$.

2. The average entanglement entropy of a random quantum state has been shown to follow the Page formula, which for large m is given approximately by
$$\bar{S} \approx \log m - \frac{1}{2} \tag{2}$$
for a bipartition of the state between two subsystems of equal Hilbert space size m. This result has been extended to definitions of the entanglement entropy based on the Rényi entropy:
$$S_\alpha \equiv \frac{1}{1-\alpha} \log\left(\sum_{i=1}^{m} s_i^{2\alpha}\right), \tag{3}$$
where $\alpha \to 1$ corresponds to the usual Von Neumann entropy definition ($S \equiv S_1$). In this general case, the expectation of S_α is
$$\bar{S}_\alpha \approx \log m - \frac{1}{\alpha - 1} \log\left(\frac{4^\alpha \Gamma(\alpha + 1/2)}{\sqrt{\pi}\Gamma(\alpha + 2)}\right) \tag{4}$$
We will verify if the entanglement entropy of random quantum states generated from different distributions agrees with the formulas above.
   (a) Prepare a code that will generate random states of the form
   $$|\Psi\rangle \propto \sum_{j,k} A_{jk} |j\rangle \otimes |k\rangle, \quad A_{jk} \equiv a_{jk} e^{i\phi_{jk}}, \tag{5}$$
   where ϕ_{jk} are uniformly distributed in $[0, \phi_{\max}]$ and a_{jk} are chosen from
      i. a uniform distribution from -1 to 1 with $\phi_{\max} = 0$.

ii. a normal distribution of variance 1 with $\phi_{\max} = 0$.

iii. a $k = 1$ chi distribution with $\phi_{\max} = 2\pi$.

iv. a $k = 2$ chi distribution with $\phi_{\max} = 2\pi$.

Normalize the states after generating them.

Suggestion: Complete the python script ex2.py. Note that you only need to generate and store the random matrix A.

(b) Calculate the entanglement spectrum by doing an SVD on A_{jk} and calculate the Rényi entropies for $\alpha = 1, 2, 3$. Is there convergence between S_α and the predictions (2) and (4)? If so, how do they converge and with what rate?

Suggestion: Calculate the error $\sigma = \left|\frac{S_\alpha - \bar{S}_\alpha}{\bar{S}_\alpha}\right|$ and average it over K random quantum states from a given distribution. Plot the error versus m. Take values of m up to 2^{10} and K possibly scaling as $\sim 1/m$ to speed up the calculation at larger m.

(c) (Optional) Study the distribution of singular values generated from random quantum states.