EPFL

Entanglement entropy

Exercise Sheet 1

1. Consider a chain of 4 qubits $|q_1q_2q_3q_4\rangle$ ($q_i = 0, 1$) in the following (unnormalized) quantum state:

$$\Psi = |0000\rangle + |0110\rangle + |1111\rangle + d(|0010\rangle + |0100\rangle + |0111\rangle + |1110\rangle - |1100\rangle - |0011\rangle). \quad (1)$$

   Taking a bipartition of the system in the middle of the chain:

   (a) Is Eq. (1) already a Schmidt decomposition for some value of d?

   (b) Calculate the entanglement spectrum and the entanglement entropy S for $d = -1, 0$ and $d \to \infty$.

2. Consider a system of N bosons (fixed number of particles) on two bosonic modes. We take a bipartition between the two modes.

   (a) Show that any state in the considered system is either a Schmidt decomposition or a product state.

   (b) If the bosons can tunnel between the wells, a simple "hopping-like" model for the system is

$$\hat{\mathcal{H}} = \sum_n |n, N-n\rangle \langle n+1, N-n-1| + \text{h.c.} \quad (2)$$

   This model is equivalent to a 1D tight-binding model and is exactly solvable. Find all the eigenstates of $\hat{\mathcal{H}}$ and their respective energies.

   (c) In the limit of large N, calculate the entanglement entropy S of all eigenstates and see that it scales as $S \sim \log N$ independently of the eigenstate.
   The following integral might be useful:

$$\int_0^1 dx \sin^2(\pi x) \log\left[\sin^2(\pi x)\right] = \frac{1}{2} - \log 2 \quad (3)$$

3. Consider a free fermion Hamiltonian $\hat{\mathcal{H}}$ on L sites partitioned into two sets, A and $\bar{A}$ ($|A| = u$). The reduced density matrix in A can be written as $\hat{\rho}_A = e^{-\hat{\mathcal{H}}_A}/Z$, where Z is determined by normalization ($\sum_i s_i^2 = 1$), and for a free fermion system the entanglement Hamiltonian $\hat{\mathcal{H}}_A$ is also a free fermion operator:

$$\hat{\mathcal{H}}_A = \sum_{l=1}^{u} \varepsilon_l \hat{\mu}_l^\dagger \hat{\mu}_l. \quad (4)$$

   Therefore, the entanglement spectrum s_i retains the free fermion nature in that it is generated from the full spectrum of the free fermionic $\hat{\mathcal{H}}_A$, whose spectrum differs from the spectrum of $\hat{\mathcal{H}}$ and is instead given by

$$\varepsilon_l = \log\left(\frac{\lambda_l}{1-\lambda_l}\right), \quad (5)$$

where λ_l are the eigenvalues of the (truncated) two-point correlation matrix:

$$C_{ij} \equiv \langle \hat{a}_i^\dagger \hat{a}_j \rangle, \quad i, j = 1, \ldots, u. \tag{6}$$

From the above statements we can then show that the entanglement entropy is given by

$$S = -\sum_{l=1}^{u} \lambda_l \log(\lambda_l) + (1 - \lambda_l) \log(1 - \lambda_l). \tag{7}$$

(a) (Optional) Prove Eq. (7).

(b) If the state of the system is particle-hole symmetric, what is the entanglement entropy of a single-site partition ($u = 1$)?