At what temperature will a hydrogen gas contain equal numbers of atoms in the ground state and in the first excited state?

$$\frac{N_n}{N_1} = \frac{g_n}{g_1} e^{-(E_n - E_1)/kT}$$

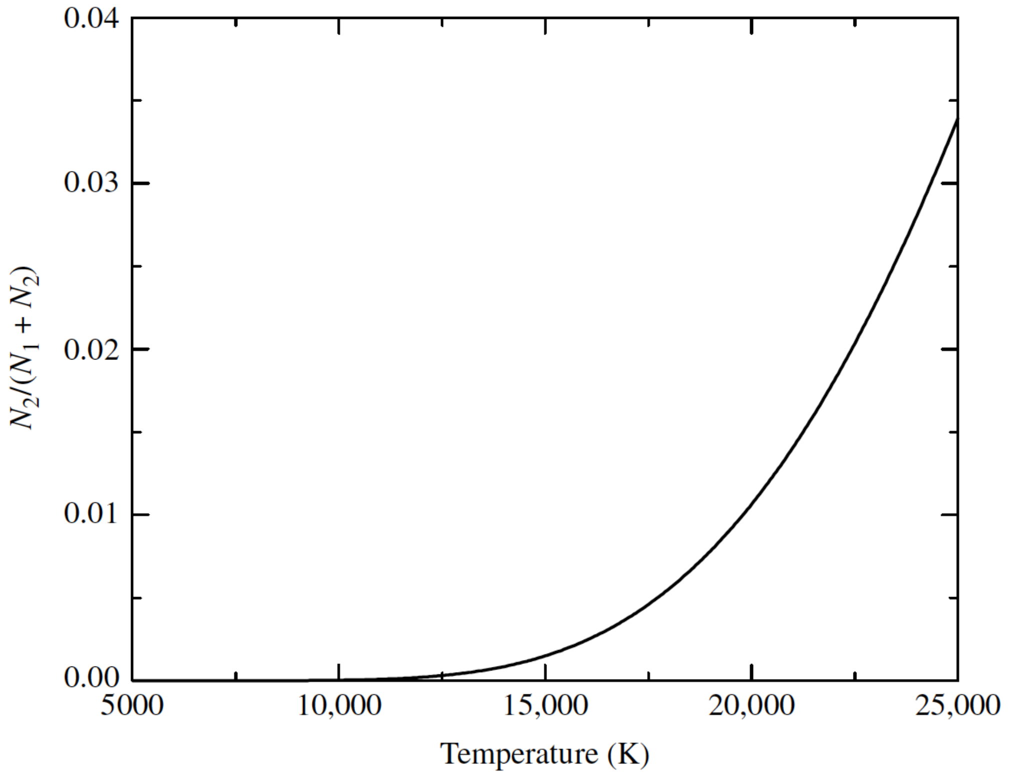
$$\frac{N_b}{N_a} = \frac{g_b\, e^{-E_b/kT}}{g_a\, e^{-E_a/kT}} = \frac{g_b}{g_a} e^{-(E_b - E_a)/kT}.$$

$$1 = \frac{2(2)^2}{2(1)^2} e^{-[(-13.6\ \text{eV}/2^2)-(-13.6\ \text{eV}/1^2)]/kT}$$

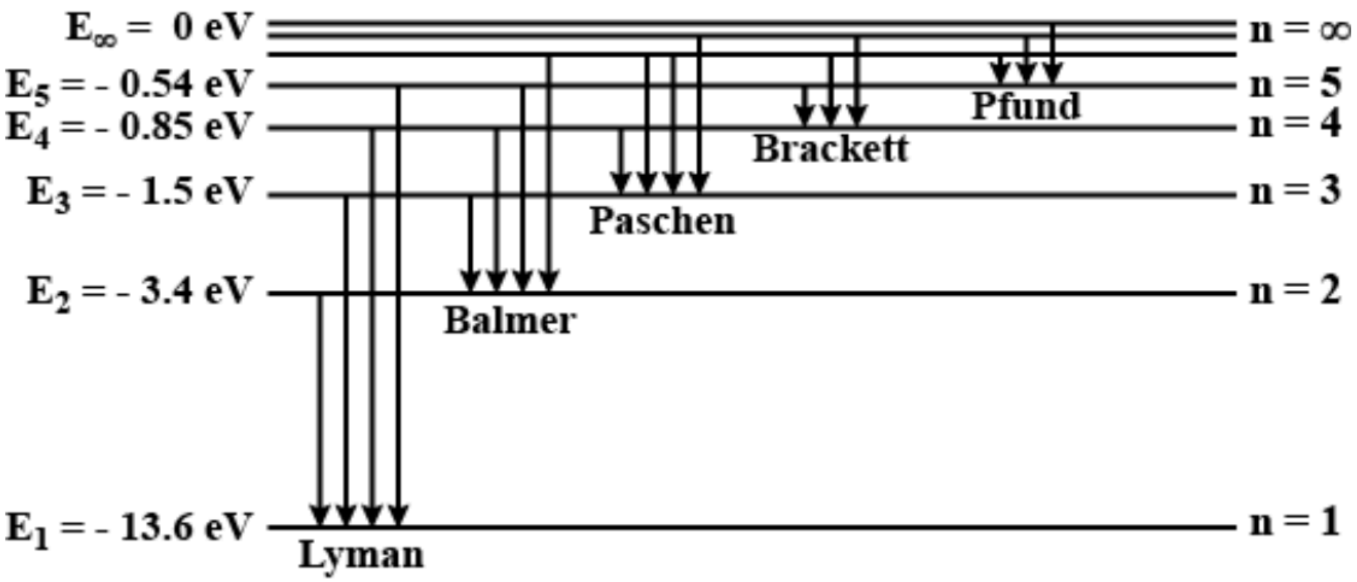
k = 8.6173423 x 10^{-5} eV K^{-1}.

$$\frac{10.2\ \text{eV}}{kT} = \ln(4)$$

$$T = \frac{10.2\ \text{eV}}{k \ln(4)} = 8.54 \times 10^4\ \text{K}$$



According to this equation, temperatures above 9520 K should lead to a higher fraction of electrons in the first excited state relative to the ground state. How, then, can we explain that the Balmer line strengths decrease at even higher temperatures?



Energy level diagram for hydrogen atom

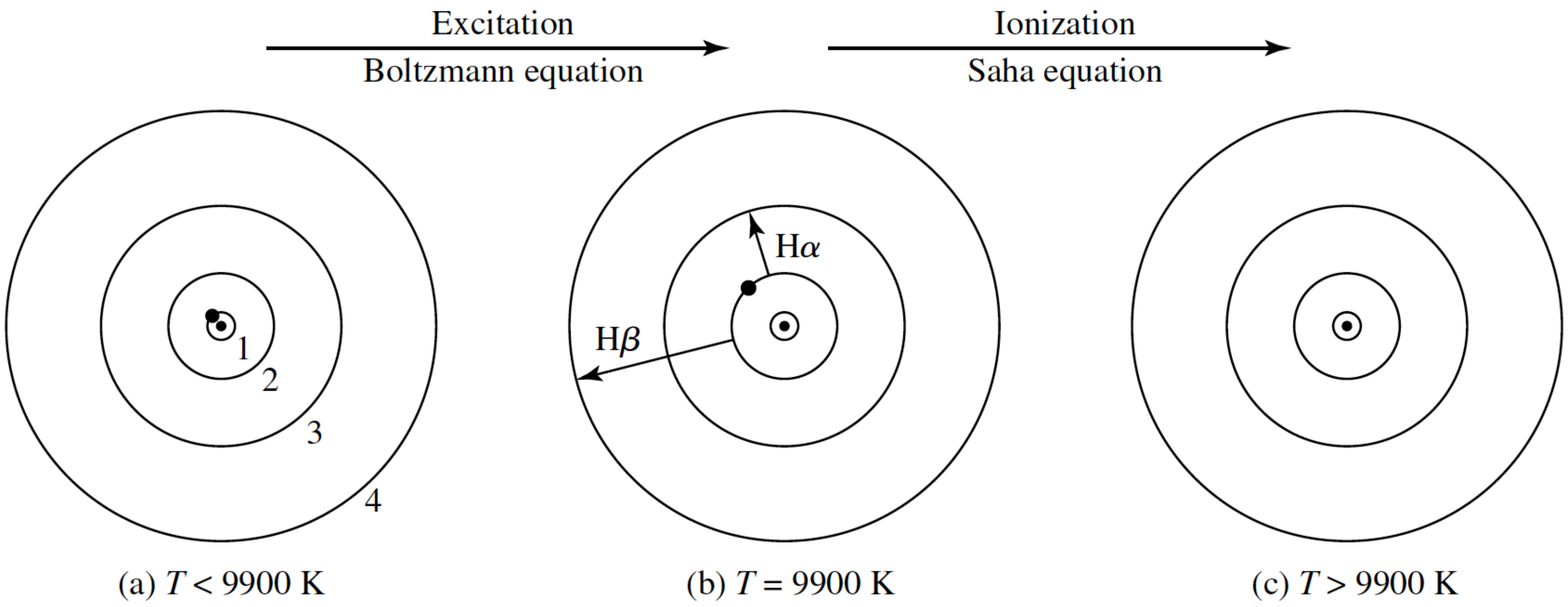
According to this equation, temperatures above 9520 K should lead to a higher fraction of electrons in the first excited state relative to the ground state. How, then, can we explain that the Balmer line strengths decrease at even higher temperatures?

One needs to consider the number of atoms in different states of ionization.

The ionization energy of hydrogen, that is, the energy required to convert H I in H II is $\chi_I = 13.6$ eV. But it is possible that the initial and final ions are not in their ground state.

We must average over the orbital energies to account for the different partitions of the electrons among the possible atomic orbitals. That is, we need to compute the partition functions of the initial and final atoms, meaning the weighted sum of the different ways the electrons can be arranged for the same energy. The higher-energy configurations (and therefore less probable) have lower weights.

$$\sum_{j=1}^{\infty} g_j \, e^{-(E_j - E_1)/kT}$$



$N_1 + N_2 \simeq N_I$ since almost all hydrogen is in its ground state

$$\frac{N_2}{N_{total}} = \left(\frac{N_2}{N_1 + N_2}\right)\left(\frac{N_I}{N_{total}}\right) = \left(\frac{N_2/N_1}{1 + N_2/N_1}\right)\left(\frac{1}{1 + N_{II}/N_I}\right)$$

We consider the degree of ionization in a stellar atmosphere that is assumed to be composed solely of hydrogen. For simplicity, we assume a constant electron pressure $P_e = 20 \text{ N m}^{-2}$.

$$\left[\frac{N_{II}}{N_I}\right]_H = \frac{2kT\,U_{II}}{P_e\,U_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT}$$

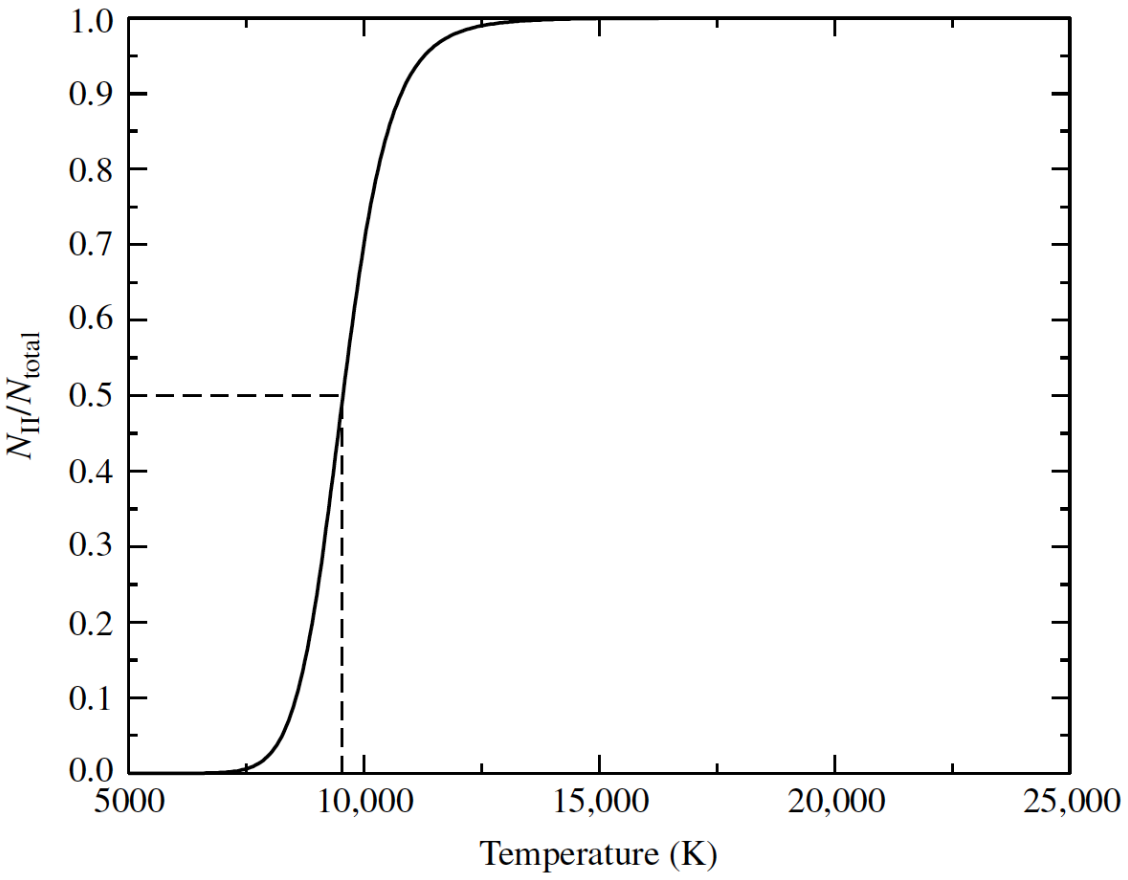
allows us to calculate the fraction of atoms that are ionized as the temperature varies from 5,000 K to 25,000 K.

$$\frac{N_{II}}{N_{total}} = \frac{N_{II}}{N_I + N_{II}} = \frac{N_{II}/N_I}{1 + N_{II}/N_I}$$

We first look at U_I and U_{II}

The hydrogen ion is just a proton and has no degeneracy, therefore. $U_{II} = 1$.
The energy of the first excited level of hydrogen is $E_2 - E_1 = 10.2$ eV above the ground level.
Since $10.2 \text{ eV} \gg kT$ for the temperatures considered, the Boltzmann factor

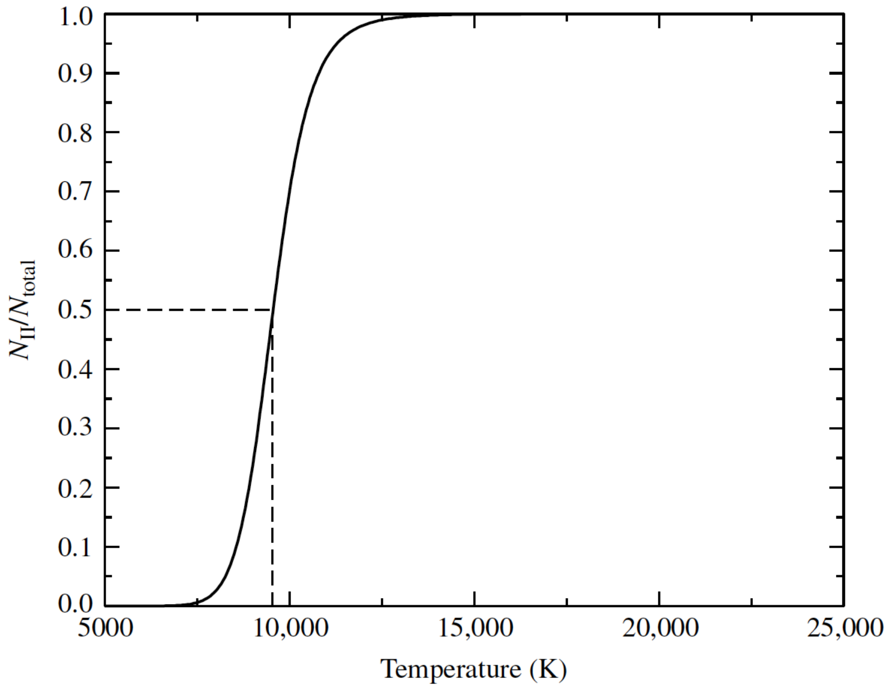
$e^{-(E_2-E_1)/kT} \ll 1$ and most of the HI atoms are in their ground state $U_I \simeq g_1 = 2$



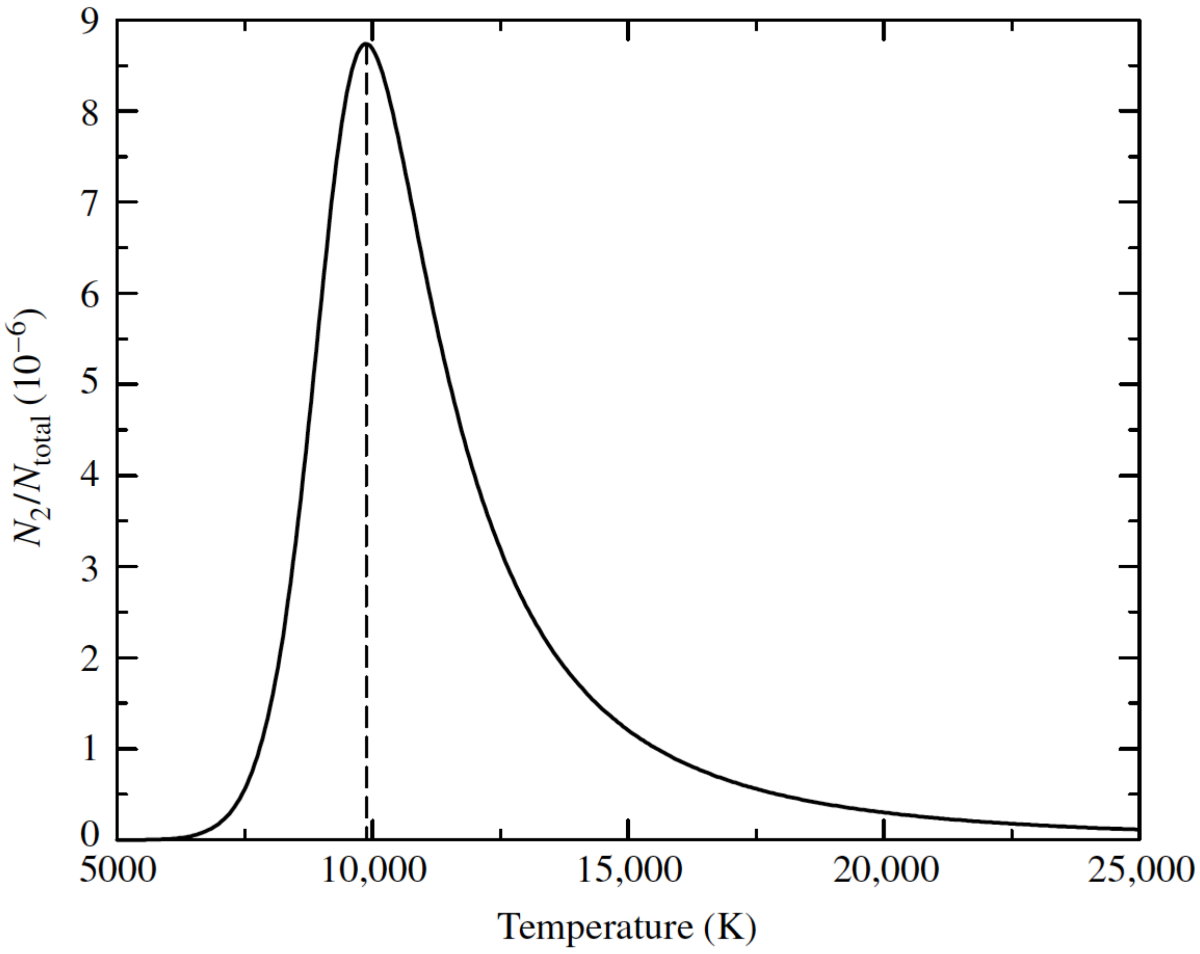
This figure shows that when T = 5000 K, essentially none of the hydrogen atoms are ionized. At about 8300 K, 5% of the atoms have become ionized. Half of the hydrogen is ionized at a temperature of 9600 K, and when T has risen to 11,300 K, all but 5% of the hydrogen is in the form of H II.

Thus the ionization of hydrogen takes place within a temperature interval of approximately 3000 K. This range of temperatures is quite limited compared to the temperatures of tens of millions of degrees routinely encountered inside stars.

The narrow region inside a star where hydrogen is partially ionized is called a hydrogen partial ionization zone and has a characteristic temperature of approximately 10,000 K for a wide range of stellar parameters.



For Pe = 20 N m^{-2}. hydrogen is 50% ionized at T≃9600K.



Hydrogen gas produces the strongest Balmer lines at a temperature of about 9900 K.
The decreasing strength of these lines at higher temperatures is due to the rapid ionization of hydrogen above 10,000 K
(that is, there are no more electrons in the second excited level!).

**"The surface of the Sun is a thin layer of the atmosphere called the photosphere. Its effective temperature is $T = 5777\ \mathrm{K}$, and there are approximately 500,000 hydrogen atoms for every calcium atom, with an electron pressure of about 1.5 N m^{-2}. Using this information, along with knowledge of statistical weights and partition functions, apply the Saha and Boltzmann equations to estimate the relative strengths of the absorption lines due to hydrogen (the Balmer lines) and the calcium lines (Ca II H and K; $\lambda = 393.3\ \mathrm{nm}$ and $396.8\ \mathrm{nm}$).

Partition functions for calcium: $u_{\mathrm{I}} = 1.32$ and $u_{\mathrm{II}} = 2.30$
Ionization energy of Ca I: $\chi = 6.11\ \mathrm{eV}$
First excited state of Ca II: $E_2 - E_1 = 3.12\ \mathrm{eV}$ above the ground state
Degeneracies of these states: $g_1 = 2$ and $g_2 = 4$."**

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We must compare the number of neutral hydrogen atoms with electrons in the first excited energy state (which produces the Balmer lines) to the number of singly ionized calcium atoms with electrons in the ground state (which produce the Ca II H and K lines).
We will use the Saha equation to determine the degree of ionization and use the Boltzmann equation to determine the distribution of electrons between the ground state and the first excited states.

Let us first consider the hydrogen atom.

Saha equation: $$\left[\frac{N_{II}}{N_I}\right]_H = \frac{2kT\,U_{II}}{P_e\,U_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT} = 7.70 \times 10^{-5} \simeq \frac{1}{13{,}000}.$$

There is one hydrogen ion (H II) for every 13,000 neutral hydrogen atoms (H I) at the surface of the Sun. Almost all of the hydrogen is neutral.

Boltzmann equation : provides how many of these neutral atoms are in the first excited state.

$g_n = 2\,n^2$ $\quad g_1 = 2$ and $g_2 = 8$

$$\left[\frac{N_2}{N_1}\right]_{H\,I} = \frac{g_2}{g_1} e^{-(E_2-E_1)/kT} = 5.06 \times 10^{-9} \simeq \frac{1}{198{,}000{,}000}$$

One hydrogen atom out of 200 million is in the first excited state and able to produce Balmer lines

$$\frac{N_2}{N_{total}} = \left(\frac{N_2}{N_1+N_2}\right)\left(\frac{N_I}{N_{total}}\right) = 5.06 \times 10^{-9}$$

Now let us consider calcium.

The Saha equation is very sensitive to the ionization energy because χ/kT it appears in the exponent and $kT \approx 0.5$ eV $\ll \chi$. Therefore, a difference of a few electron-volts in the ionization energy produces a change of several orders of magnitude in the Saha equation..

$$\left[\frac{N_{\rm II}}{N_{\rm I}}\right]_{\rm Ca} = \frac{2kT\, u_{i+1}}{P_e\, u_i}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_{\rm I}/kT} = 918.$$

Practically all calcium atoms are in the form of Ca II; only about one atom out of ~900 remains neutral.

We will use the Boltzmann equation to estimate how many of these calcium ions are in the ground state, capable of forming the Ca II H and K absorption lines (we do not distinguish between these two lines – we treat them as if they were the same).

$$\left[\frac{N_2}{N_1}\right]_{\rm Ca\ II} = \frac{g_2}{g_1} e^{-(E_2-E_1)/kT} = 3.79 \times 10^{-3} = \frac{1}{264}$$

Out of 265 Ca II ions, all but one are in the ground state and capable of producing the Ca II K line
⇒ all calcium atoms in the Sun's photosphere are capable of producing the Ca II H and K lines.

$$\left[\frac{N_1}{N_{\text{total}}}\right]_{\text{Ca II}} \simeq \left[\frac{N_1}{N_1 + N_2}\right]_{\text{Ca II}} \left[\frac{N_{\text{II}}}{N_{\text{total}}}\right]_{\text{Ca}}$$

$$= \left(\frac{1}{1 + [N_2/N_1]_{\text{Ca II}}}\right) \left(\frac{[N_{\text{II}}/N_{\text{I}}]_{\text{Ca}}}{1 + [N_{\text{II}}/N_{\text{I}}]_{\text{Ca}}}\right)$$

$$= \left(\frac{1}{1 + 3.79 \times 10^{-3}}\right) \left(\frac{918}{1 + 918}\right)$$

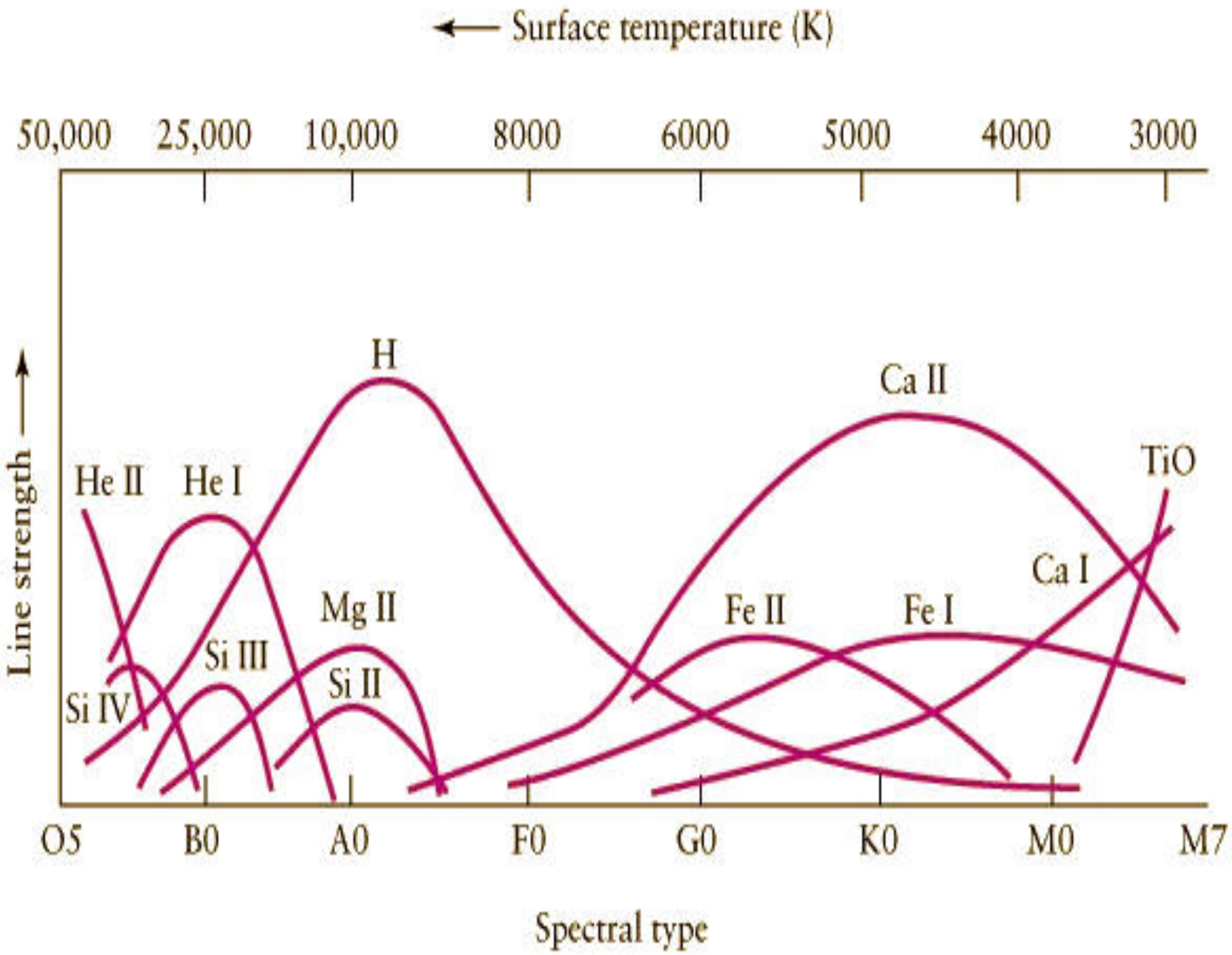
$$= 0.995.$$

We can therefore understand why the Ca II H and K lines are much stronger in the Sun than the Balmer lines. There are 500,000 hydrogen atoms for every calcium atom, but only a very small fraction, 5.06×10^{-9} , of these hydrogen atoms are capable of producing a Balmer line.

By multiplying these two factors,$(500{,}000) \times (5.06 \times 10^{-9}) \approx 0.00253 = 1/395$, we see that there are approximately 400 times more Ca II ions with electrons in the ground state than neutral hydrogen atoms with an electron in the first excited state.

The strength of the H and K lines is not due to a greater abundance of calcium, but reflects the temperature sensitivity of the excitation and ionization states.

Effect of temperature in the Saha and Boltzmann equations: ionization as a function of spectral type.



Ionisation de H et de He

(source:R. Bowers & T. Deeming,
« Astrophysics I: Stars », Joes and Bartlett
Publishers, 1984)

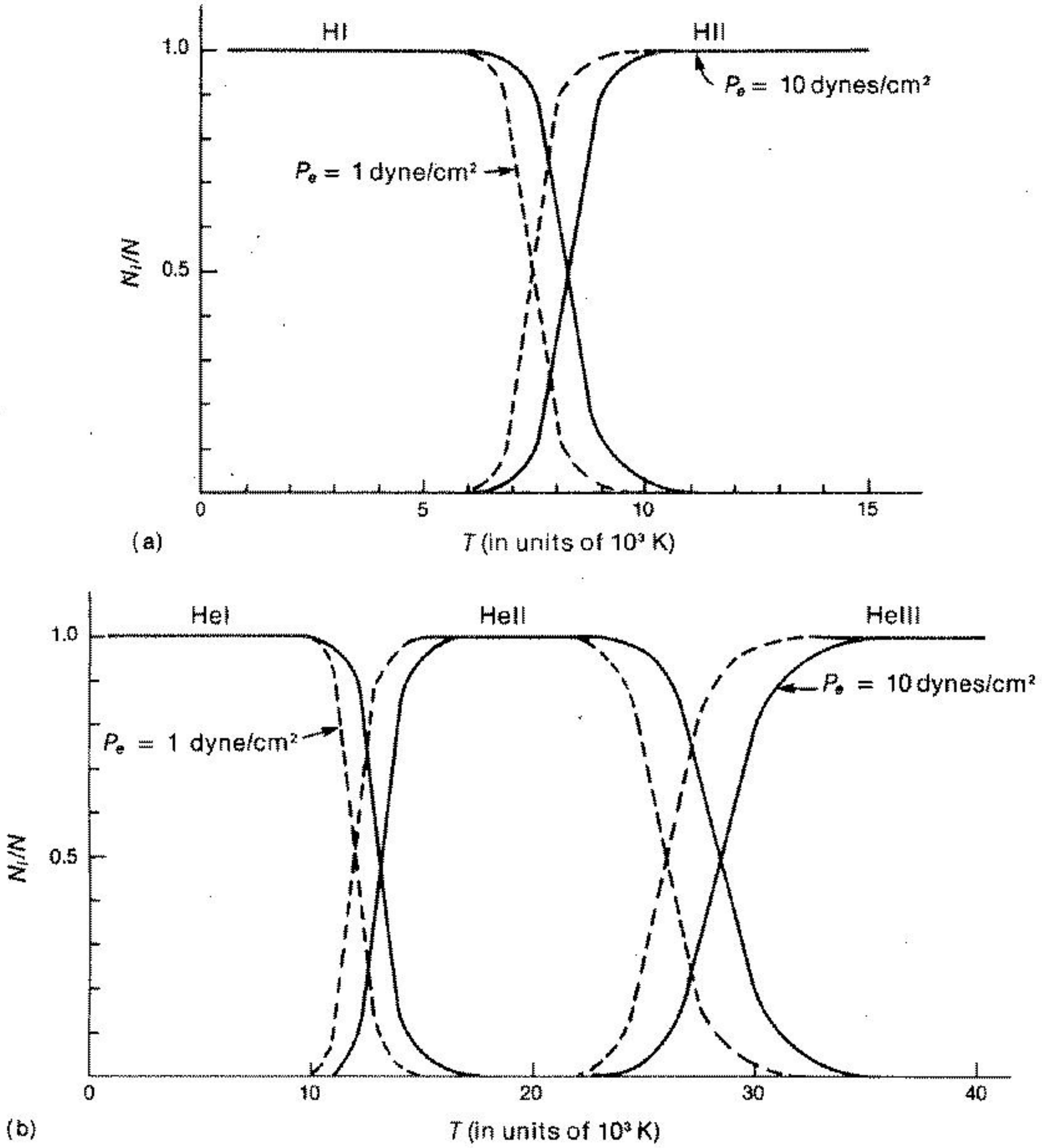


Figure 6.3. (a) Fractional ionization of hydrogen versus temperature (solid curve is electron pressure $P_e = 10$ dynes/cm^2; dashed curve is $P_e = 1$ dyne/cm^2). (b) Fractional ionization of He (for $P_e = 1$ and $P_e = 10$ dynes/cm^2).