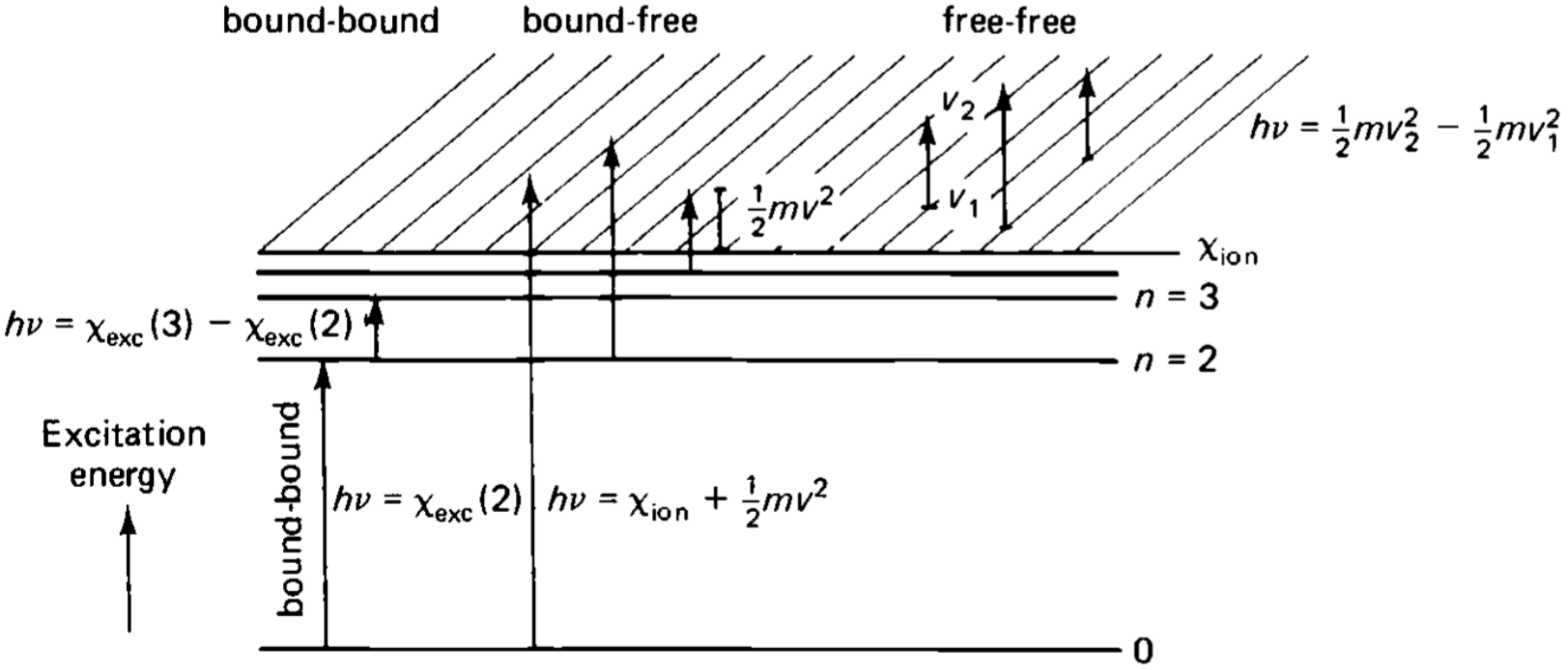
$$\kappa_\nu = [\kappa_{\mathrm{bf}}(\nu) + \kappa_{\mathrm{ff}}(\nu) + \kappa_{\mathrm{bb}}(\nu) + \cdots]\left(1 - e^{-h\nu/kT}\right) + \kappa_{\mathrm{es}} + \cdots$$



Although the theory of stellar opacity is complex in detail, the end results are easily summarized, with important consequences for the macroscopic properties of stars.

Except the electron scattering κes , the other sources of opacity depend on the frequency of the emitted photon ν

Absorption or emission bound-free

Possible transitions for all atoms or ions that still have a bound electron.

The frequency of the absorbed photon is given by if the electron is taken out of level n

$$h\nu = \chi_n + 1/2mv^2$$

A bound–free transition from level n requires a minimum energy.

$$\chi_n = \chi_{ion} - \chi_{exc}(n)$$

ionization energy

Transition bound-bound

Transitions between atomic levels — about 10% in the solar core, and 50% in the outer regions (Millions of transitions must be taken into account).

Transition bound-free

Photoionization occurs when the energy of an incident photon exceeds the binding energy of an electron.

$$\kappa_{\mathrm{bf}} = \kappa_{\mathrm{bf}}^{(0)}\,\rho\,T^{-3.5}, \quad \text{with} \quad \kappa_{\mathrm{bf}}^{(0)} \simeq 4.3 \times 10^{25}\,Z\,(1+X),$$

Kramer’s Law

Depends on the star’s metallicity → the main contributor for low-mass stars such as the Sun.

Transition free-free

Within the Coulomb field of an ion, an electron may emit (bremsstrahlung) or absorb (free–free) a photon. It is the dominant process in metal-poor stars.

$$\kappa_{\mathrm{ff}} = \kappa_{\mathrm{ff}}^{(0)}\,\rho\,T^{-3.5}, \quad \text{with} \quad \kappa_{\mathrm{ff}}^{(0)} \simeq 3.7 \times 10^{22}(X+Y)(1+X),$$

Electron scattering

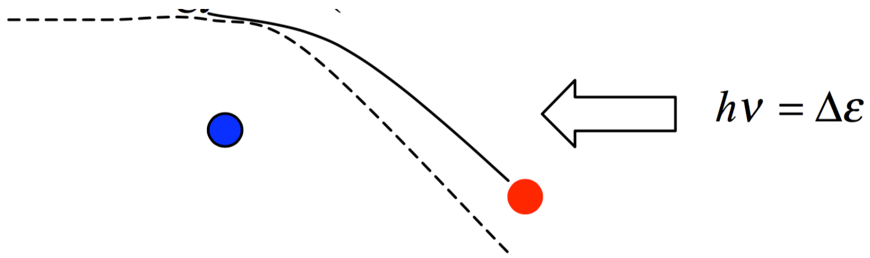
This process dominates in massive stars. Electrons are unable to absorb photons but can scatter them, resulting in a decrease in flux. The opacity remains nearly constant.

$$\kappa_{\mathrm{es}} = 0.200\,(1 + X),$$

Free-free emission = Bremsstrahlung (German for ‘braking radiation’)

Un électron en présence d’un ion fait la transition d’un état libre (ionisé) à un autre état libre

Contrepartie en absorption: l’électron fait la transition d’un état d’énergie positive à un état d’énergie plus élevée, en absorbant un photon du continu du champ de radiation



$$\kappa \approx C\rho T^{-7/2}$$

$$\rho\kappa \propto \rho^2$$

pour des rayonnements à
2 particules
matérielles

rayonnement ff diminue
avec T

Bound–free absorption coefficient for a hydrogenic atom

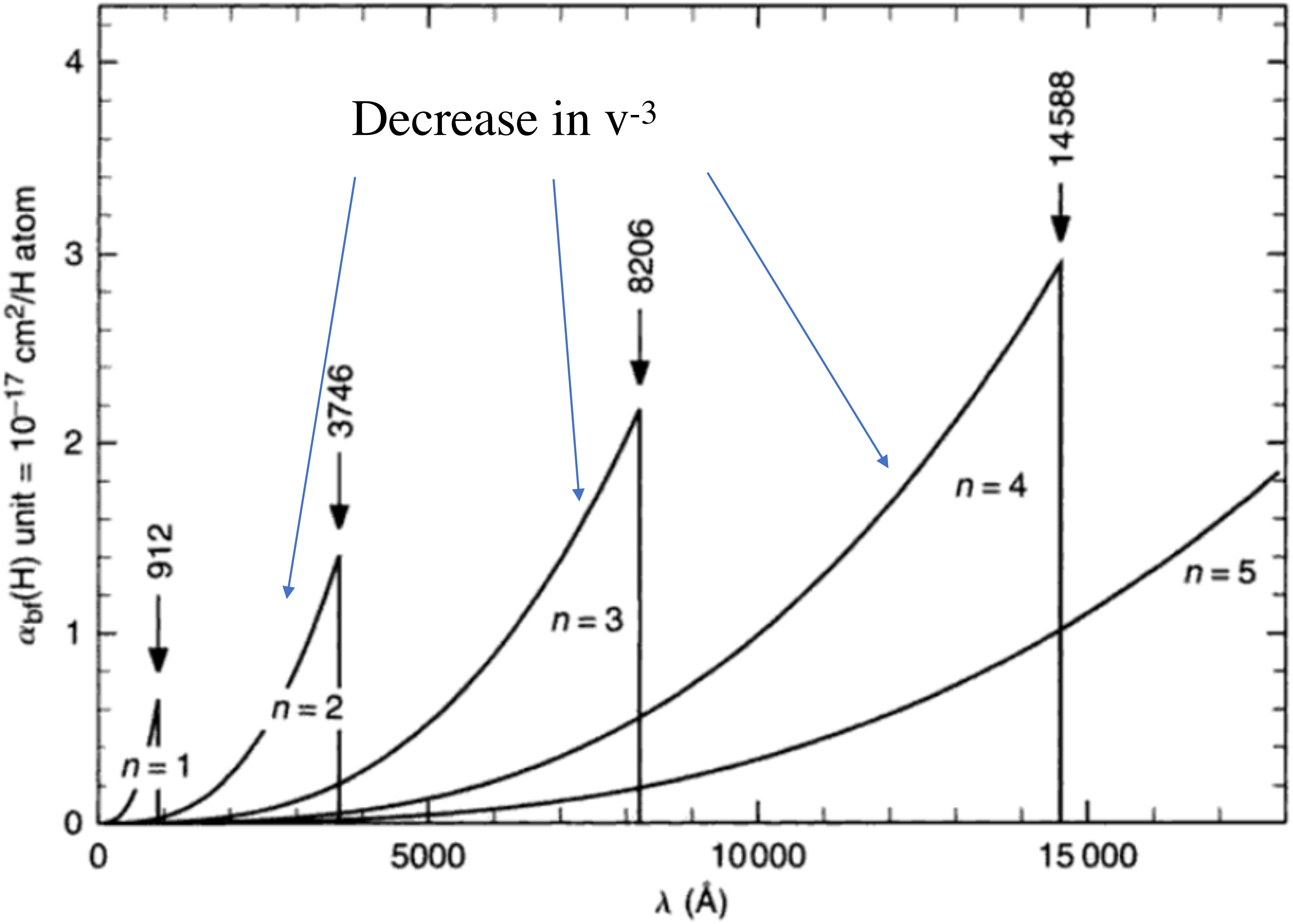
Frequency of photon ν
Electron on level n
Ion/nucleus charge Zj
For hydrogène Zj =1.

$$\boxed{\alpha_{bf} = \frac{64\,\pi^4\, m_e\, e^{10}}{3^{3/2}\, c\, h^6} \quad \frac{Z_j^4}{n^5\,\nu^3}\, g_{bf}}$$

g_{bf} = Gaunt factor , quantum correction of the absorption coefficient calculated in classical physics, of a few percent for the hydrogen atom, often poorly known and important for other atoms.

$$\begin{aligned}\alpha_{\rm bf}(\mathrm{H}) &= \frac{32}{3^{3/2}}\frac{\pi^2 e^6}{h^3}\frac{R}{n^5\nu^3}g_{\rm bf}\\ &= \frac{32}{3^{3/2}}\frac{\pi^2 e^6}{h^3 c^3}R\frac{\lambda^3}{n^5}g_{\rm bf}\\ &= \alpha_0 g_{\rm bf}\frac{\lambda^3}{n^5}\ \mathrm{cm}^2 \text{ per neutral H atom}\end{aligned}$$

$\alpha_0 = 1.0449 \times 10^{-26}$ for λ in angstroms.



Decrease in ν^{-3}

Fig. 8.2. The bound–free absorption coefficient for hydrogen increases with n.

Schematic representation of the evolution of the extinction coefficient α resulting from bound–free absorption processes.

Each excitation level of the hydrogen atom produces a step.

Similar features are observed in stellar spectra, not only for hydrogen but also for other elements, which undergo comparable processes.

Bound Free Continua

• Lyman	*far UV*	n=1
• Balmer	*UV*	n=2
• Paschen	*optical*	n=3
• Bracket	*IR*	n=4

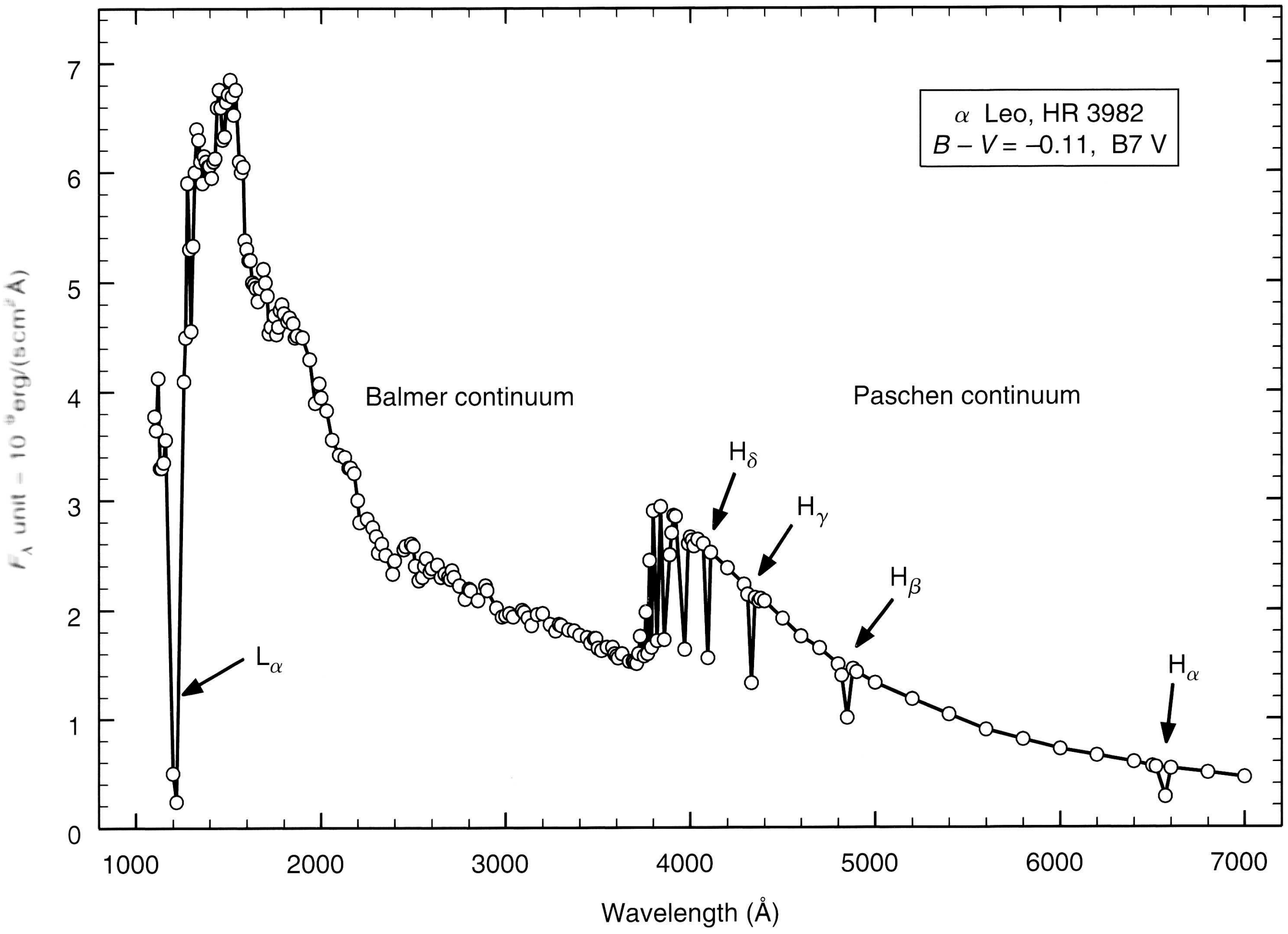


Fig. 10.7. The energy distribution of α Leo is strongest in the ultraviolet, and this portion of the spectrum is therefore very valuable in modeling the star. Even with low resolution, the hydrogen lines are prominent. Data from Code *et al*. (1976).

Opacity due to H^- ions

If the temperature drops below 10^4–10^5 K, H (and He) recombine to form neutral atoms, and the number of free electrons available for free–free absorption decreases, as does the number of photons (which decreases exponentially with T) available for bound–free absorption. Nevertheless, the Sun’s atmosphere becomes rapidly opaque at ~5800 K!

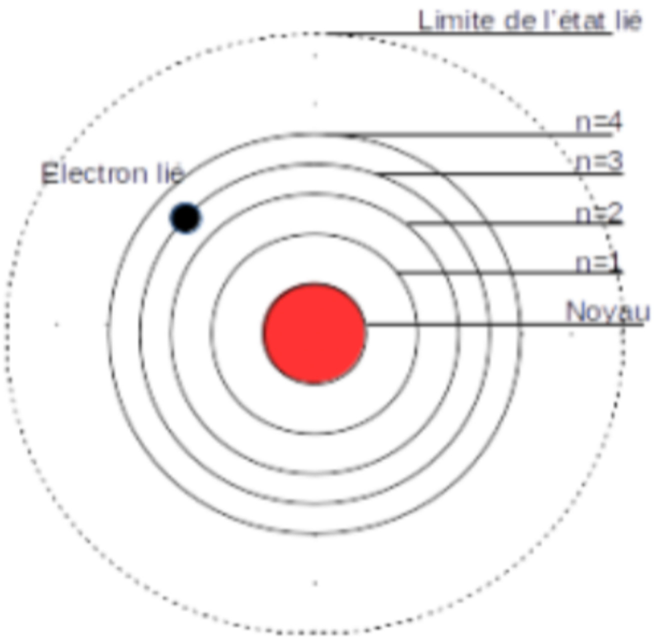
Rupert Wildt (1905–1976) provided the correct solution in 1939, and Subrahmanyan Chandrasekhar (1910–1995) carried out the quantum calculation.
For solar-type and cooler stars, the opacity is dominated by H^-.
H^- has one bound state and an infinite number of ionized states. Its ionization potential is 0.754 eV, which is very low.
Photoionisation (bound–free) occurs easily:

$$H^- \rightarrow H + e^-$$

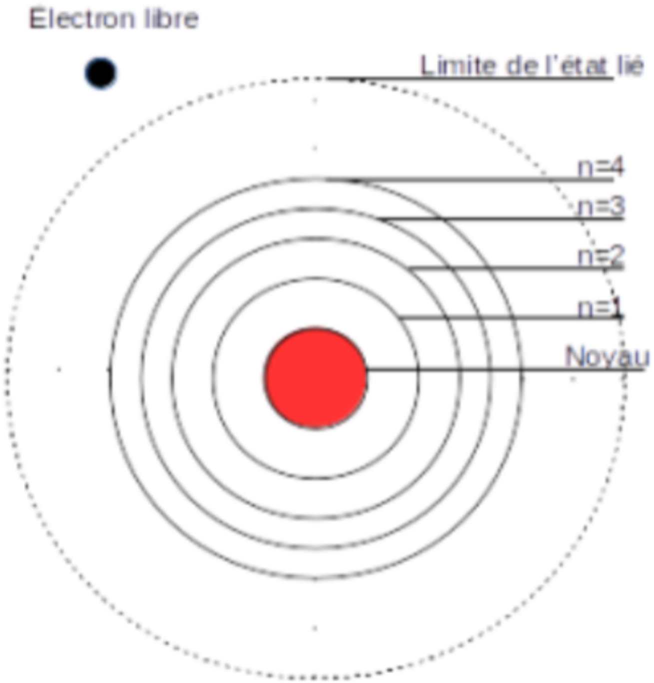
$$H^- \rightarrow H + e^-$$

Free–free absorption is also possible due to interactions of H^- with other free electrons

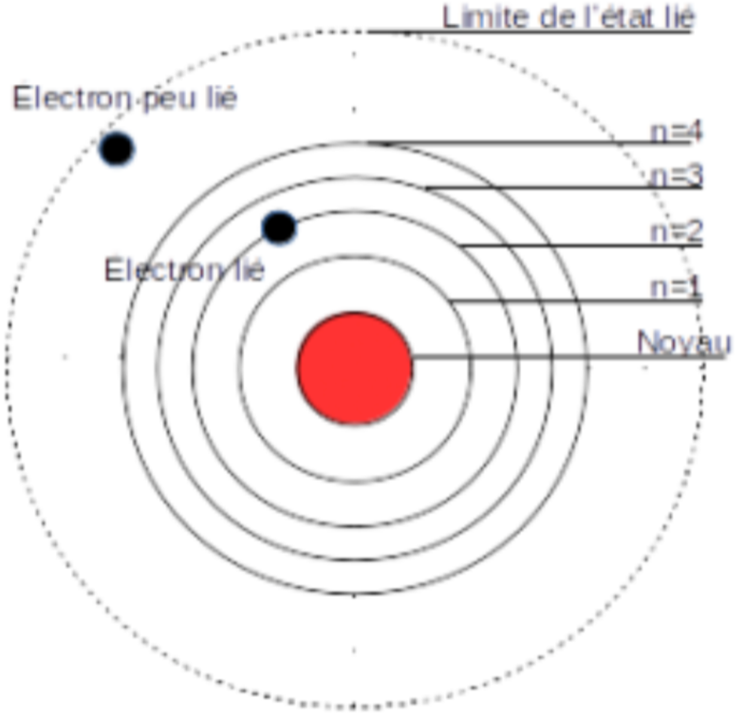
Opacity due to ions H^-



Hydrogène neutre (HI)

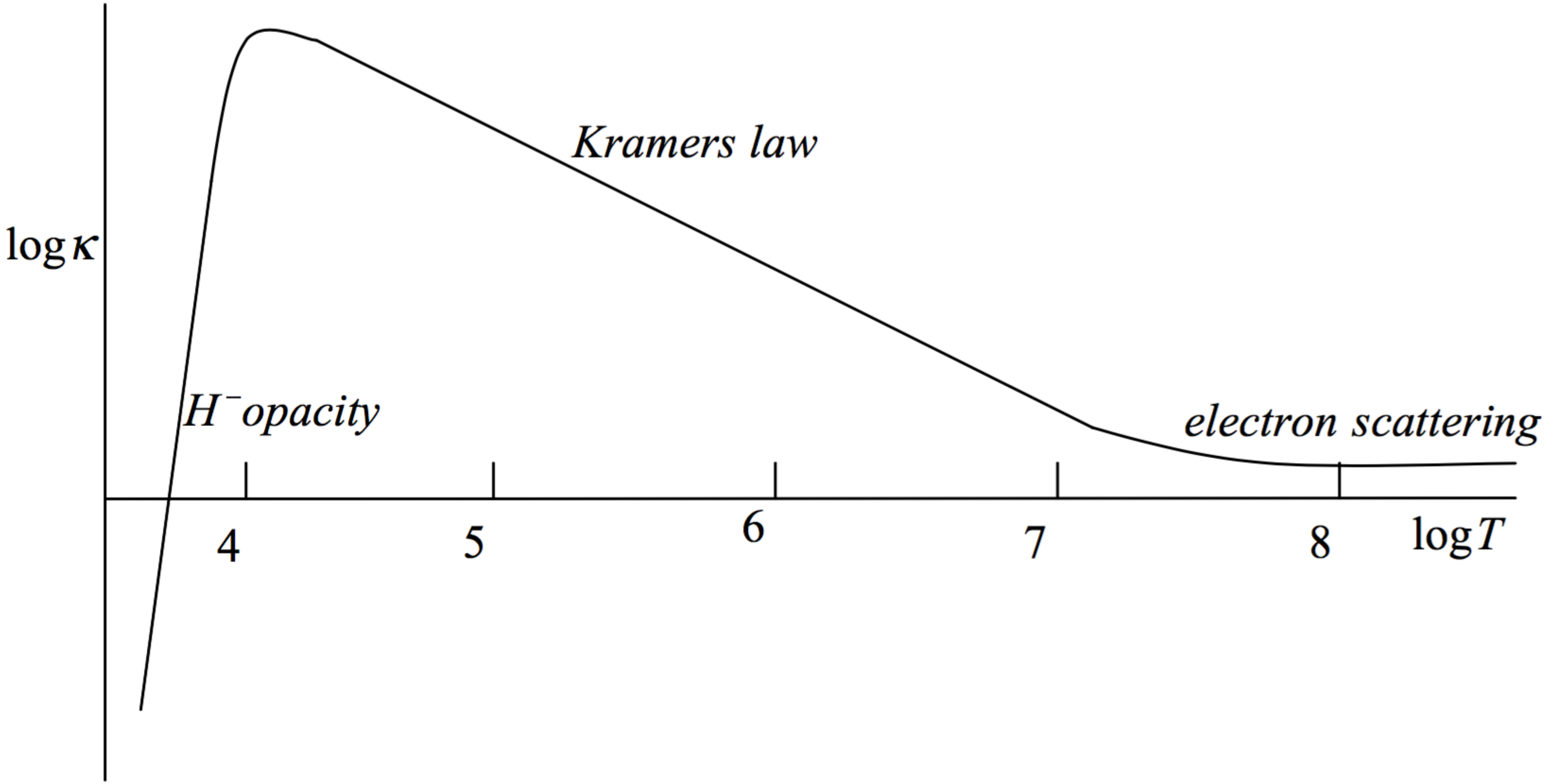


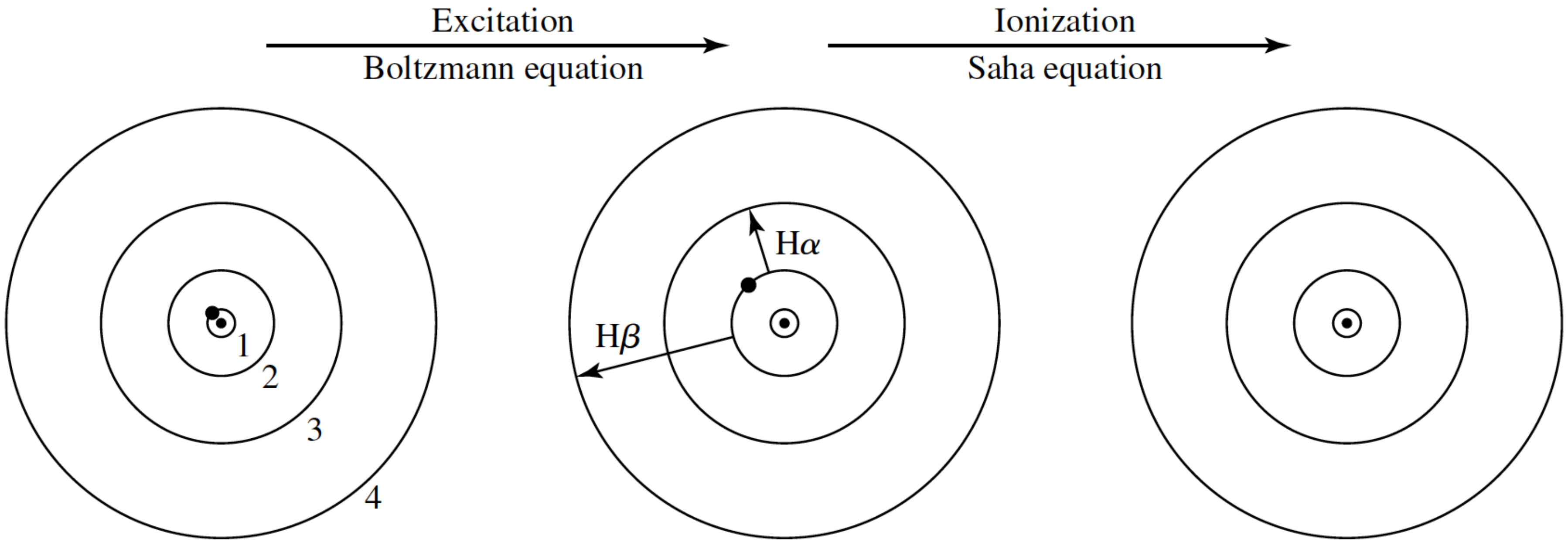
Hydrogène ionisé (HII)



Ion H^-

$$H + e^- \rightleftharpoons H^-$$


log κ
Kramers law
H⁻ opacity
electron scattering
4
5
6
7
8
log T

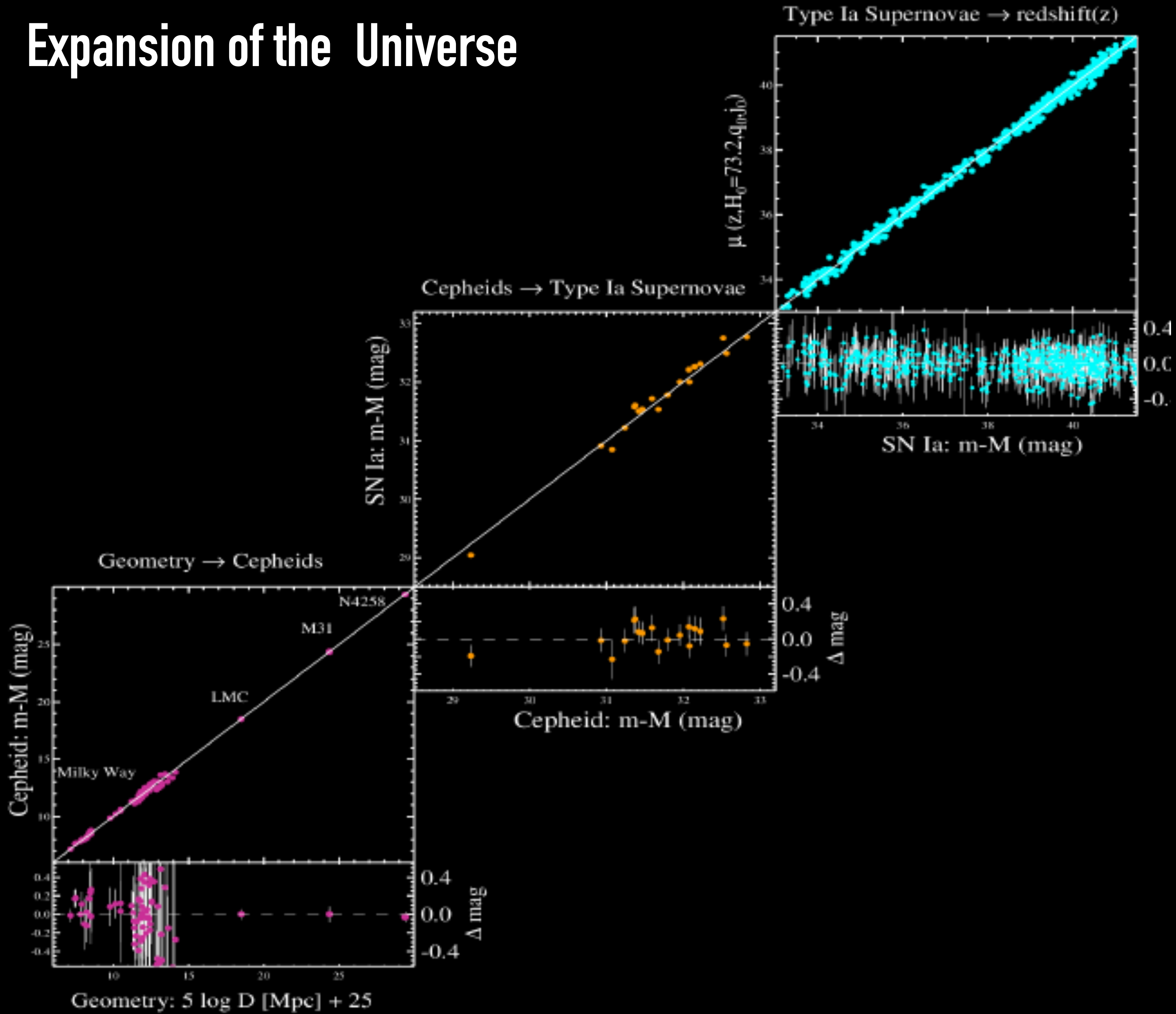


(a) $T < 9900$ K

(b) $T = 9900$ K

(c) $T > 9900$ K

Expansion of the Universe



Cepheids:

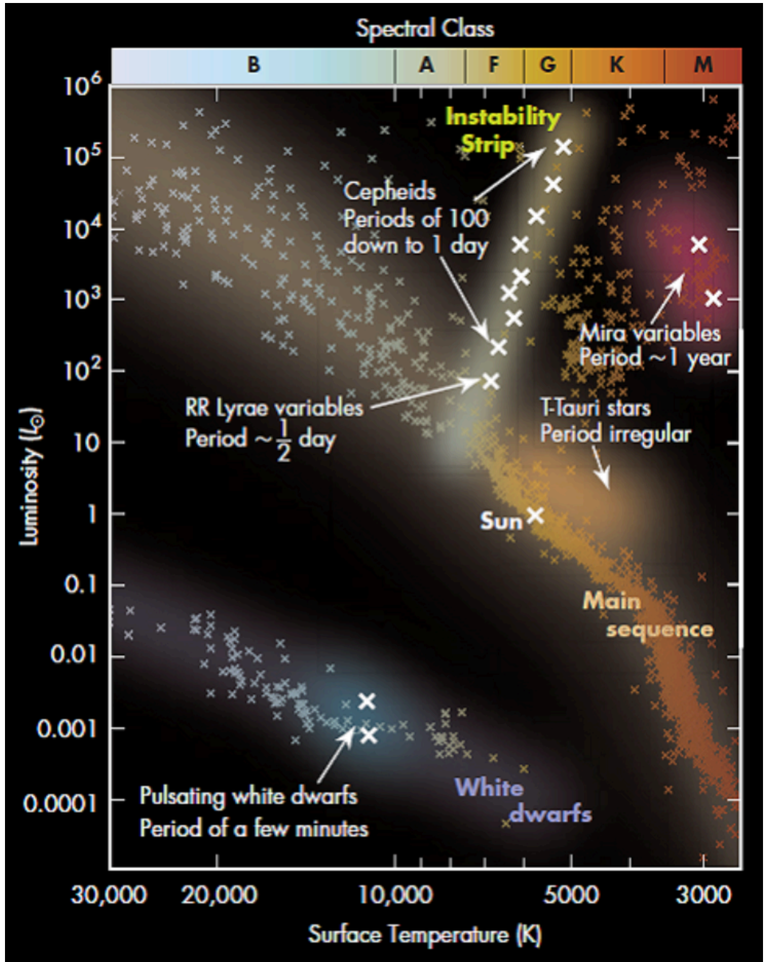
50–300 Myr

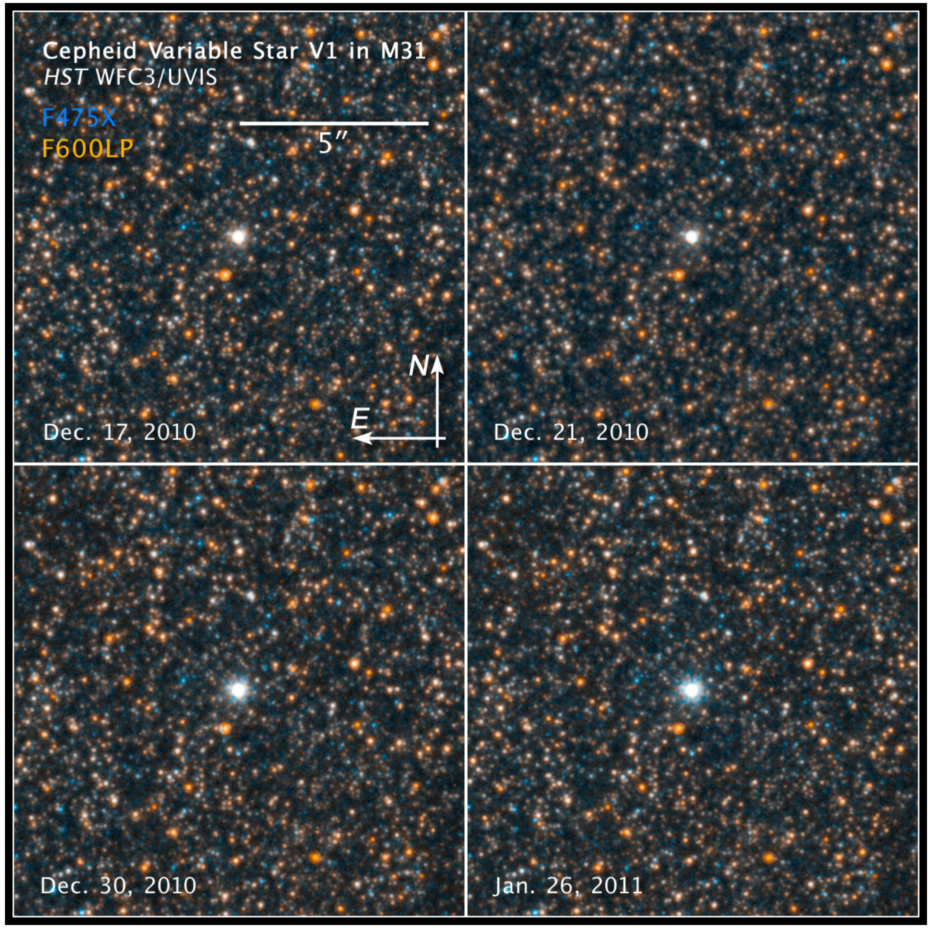
Intermediate masses (4–10 $M_\odot$)

Luminous (~10^3–$10^5 L_\odot$)

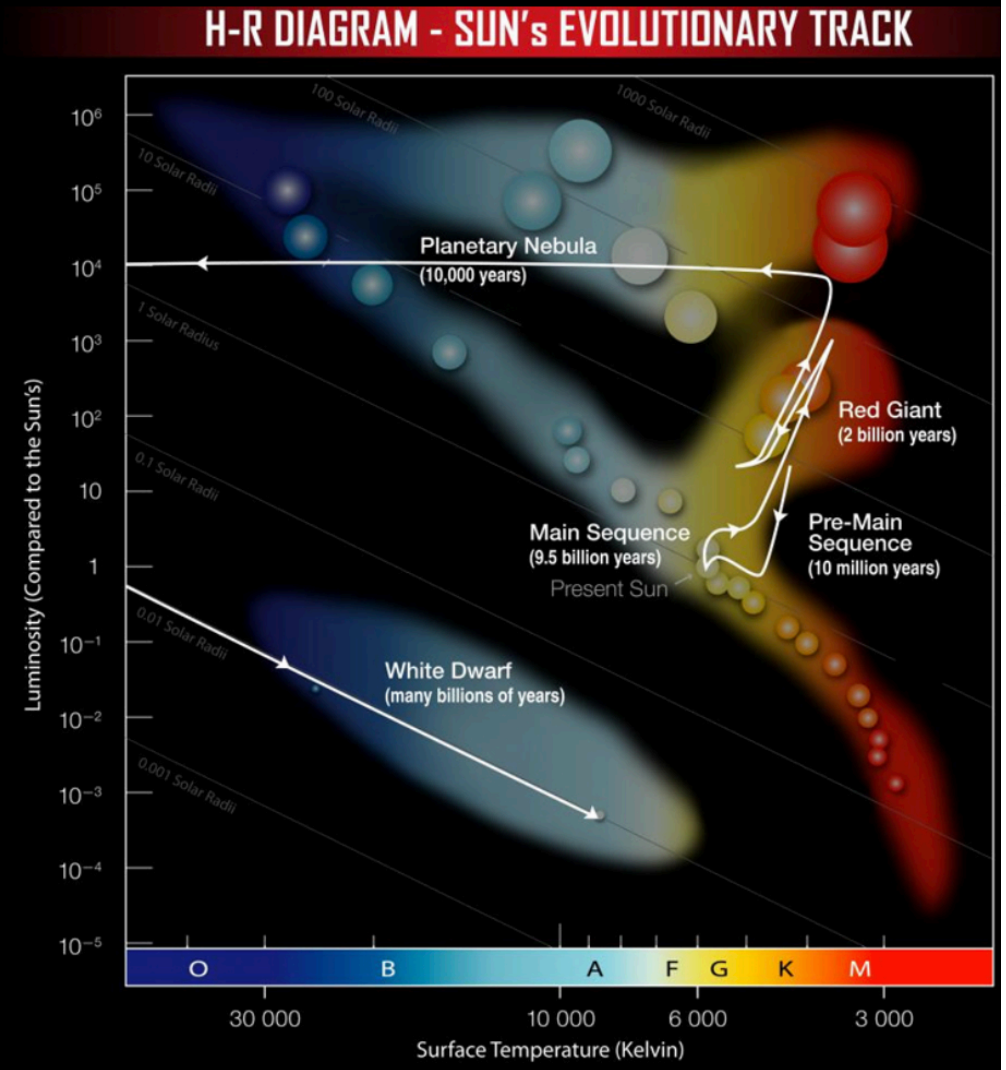
F6–K2 supergiants whose radial pulsations produce variations in radius, temperature, and brightness

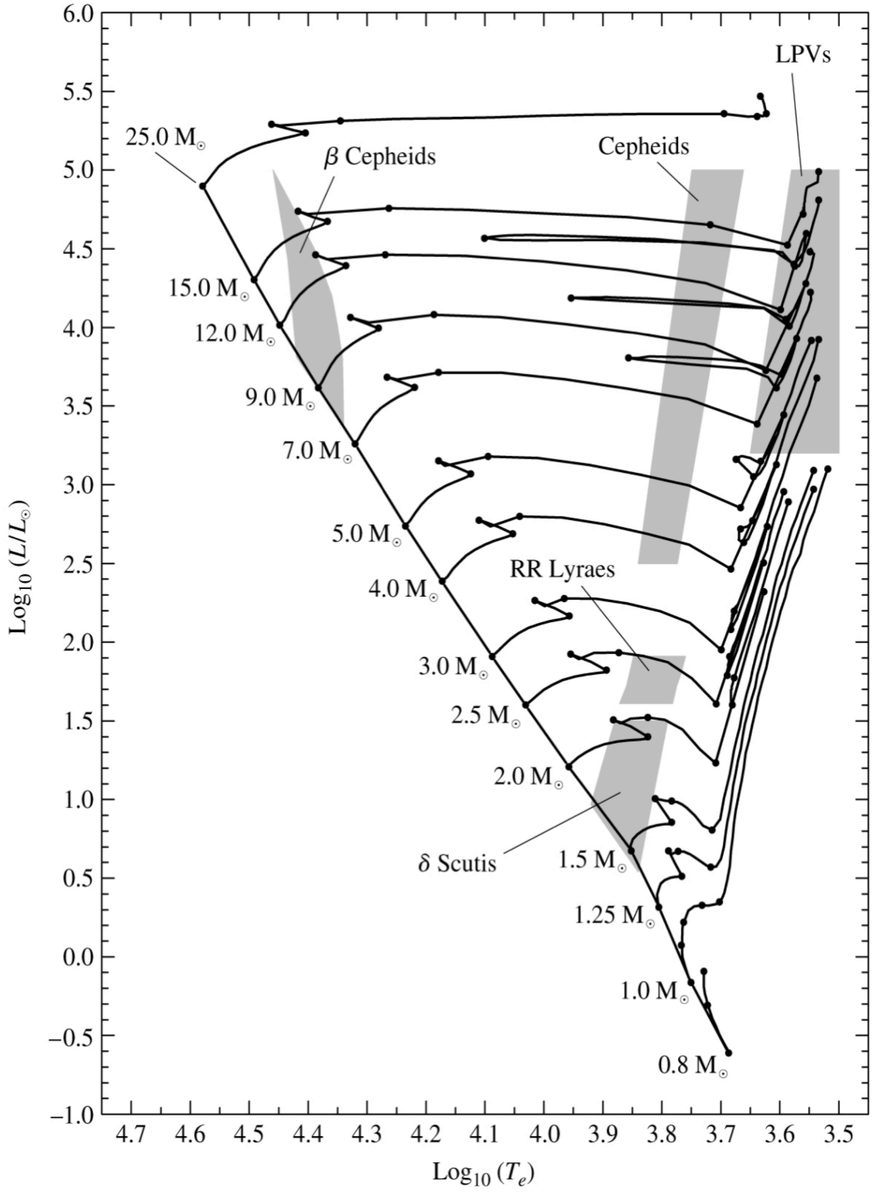
Pulsation periods range from 1.5 days to ~45 days.



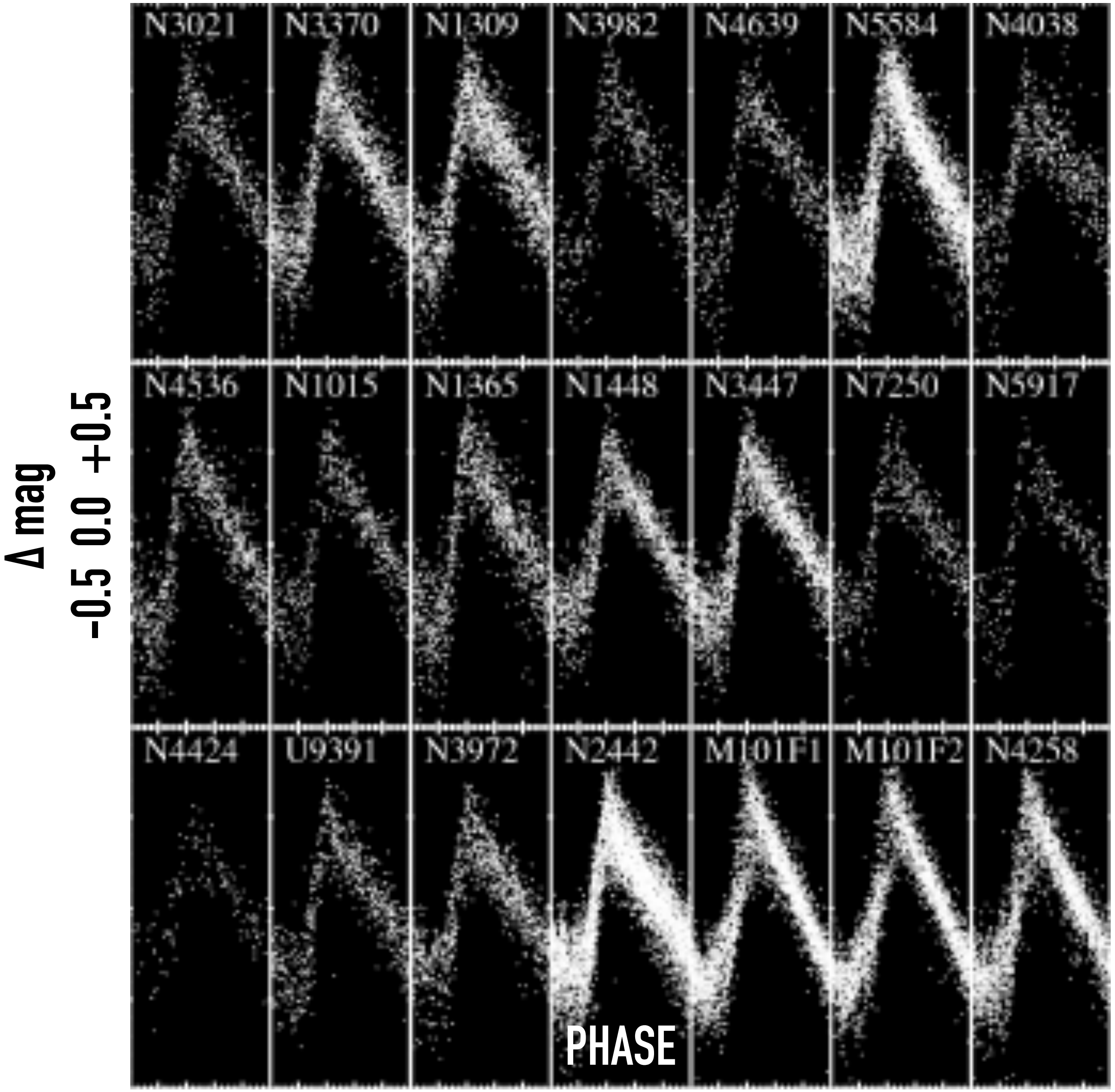


Cepheids


H-R DIAGRAM - SUN's EVOLUTIONARY TRACK
Luminosity (Compared to the Sun's)
10^6
10^5
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
100 Solar Radii
1000 Solar Radii
10 Solar Radii
1 Solar Radius
0.1 Solar Radii
0.01 Solar Radii
0.001 Solar Radii
Planetary Nebula
(10,000 years)
Red Giant
(2 billion years)
Main Sequence
(9.5 billion years)
Pre-Main Sequence
(10 million years)
Present Sun
White Dwarf
(many billions of years)
O
B
A
F
G
K
M
30 000
10 000
6 000
3 000
Surface Temperature (Kelvin)

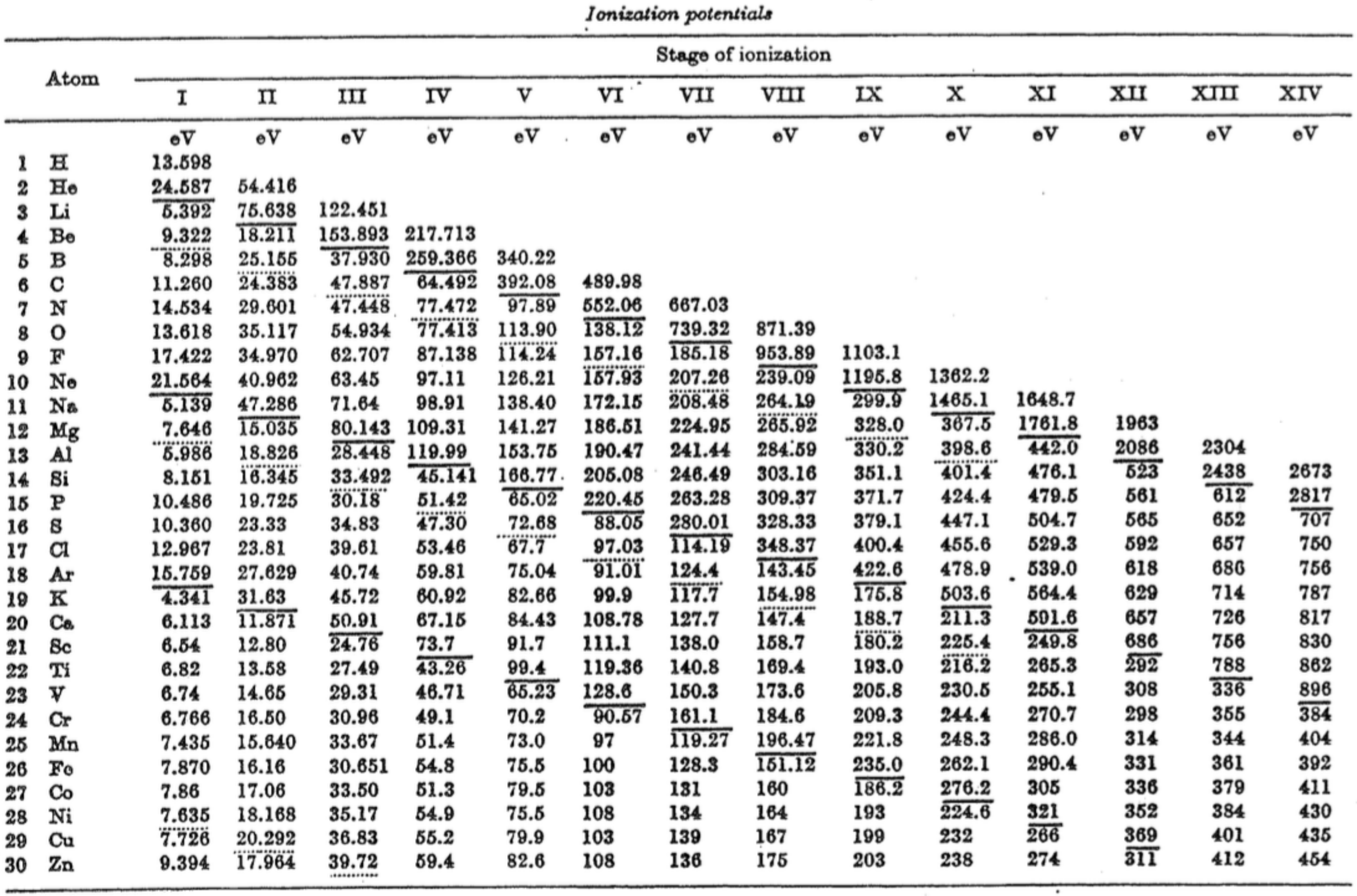

$\text{Log}_{10}(L/L_\odot)$
6.0
5.5
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
LPVs
Cepheids
β Cepheids
RR Lyraes
δ Scutis
25.0 $M_\odot$
15.0 $M_\odot$
12.0 $M_\odot$
9.0 $M_\odot$
7.0 $M_\odot$
5.0 $M_\odot$
4.0 $M_\odot$
3.0 $M_\odot$
2.5 $M_\odot$
2.0 $M_\odot$
1.5 $M_\odot$
1.25 $M_\odot$
1.0 $M_\odot$
0.8 $M_\odot$
4.7
4.6
4.5
4.4
4.3
4.2
4.1
4.0
3.9
3.8
3.7
3.6
3.5
$\text{Log}_{10}(T_e)$

In partially ionized zones, compression increases ionization rather than temperature.



During compression, the opacity rises, causing energy to be trapped in the ionization zone. The resulting increase in pressure drives expansion, which in turn lowers the opacity and allows the release of the stored pressure and energy. Following this release, gravitational contraction resumes, restarting the cycle.

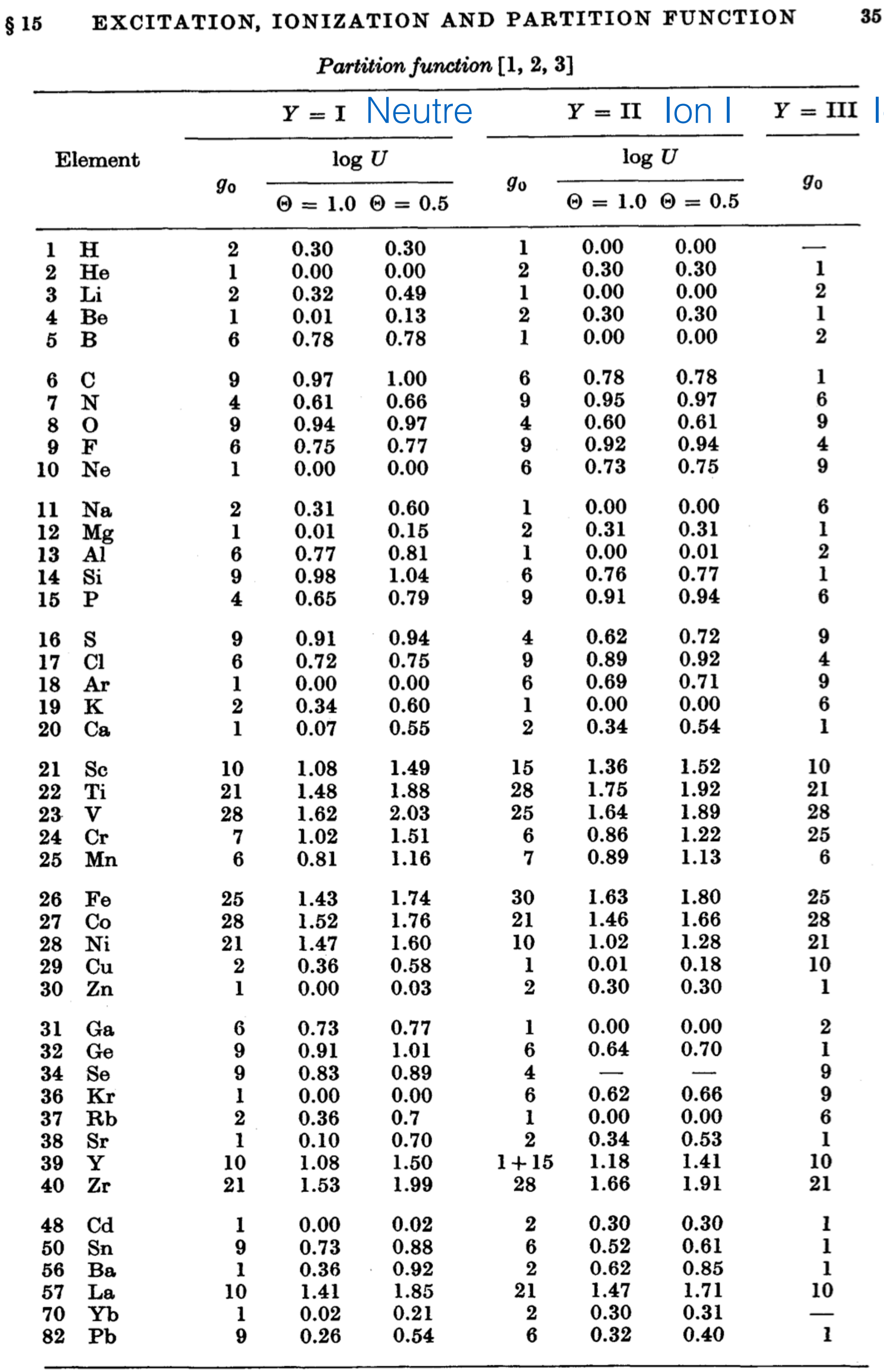
The Saha equation involves the excitation potentials.



Ionization potentials

	Atom	Stage of ionization													
		I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV
		eV	eV	eV	eV	eV	eV	eV	eV	eV	eV	eV	eV	eV	eV
1	H	13.598													
2	He	24.587	54.416												
3	Li	5.392	75.638	122.451											
4	Be	9.322	18.211	153.893	217.713										
5	B	8.298	25.155	37.930	259.366	340.22									
6	C	11.260	24.383	47.887	64.492	392.08	489.98								
7	N	14.534	29.601	47.448	77.472	97.89	552.06	667.03							
8	O	13.618	35.117	54.934	77.413	113.90	138.12	739.32	871.39						
9	F	17.422	34.970	62.707	87.138	114.24	157.16	185.18	953.89	1103.1					
10	Ne	21.564	40.962	63.45	97.11	126.21	157.93	207.26	239.09	1195.8	1362.2				
11	Na	5.139	47.286	71.64	98.91	138.40	172.15	208.48	264.19	299.9	1465.1	1648.7			
12	Mg	7.646	15.035	80.143	109.31	141.27	186.51	224.95	265.92	328.0	367.5	1761.8	1963		
13	Al	5.986	18.826	28.448	119.99	153.75	190.47	241.44	284.59	330.2	398.6	442.0	2086	2304	
14	Si	8.151	16.345	33.492	45.141	166.77	205.08	246.49	303.16	351.1	401.4	476.1	523	2438	2673
15	P	10.486	19.725	30.18	51.42	65.02	220.45	263.28	309.37	371.7	424.4	479.5	561	612	2817
16	S	10.360	23.33	34.83	47.30	72.68	88.05	280.01	328.33	379.1	447.1	504.7	565	652	707
17	Cl	12.967	23.81	39.61	53.46	67.7	97.03	114.19	348.37	400.4	455.6	529.3	592	657	750
18	Ar	15.759	27.629	40.74	59.81	75.04	91.01	124.4	143.45	422.6	478.9	539.0	618	686	756
19	K	4.341	31.63	45.72	60.92	82.66	99.9	117.7	154.98	175.8	503.6	564.4	629	714	787
20	Ca	6.113	11.871	50.91	67.15	84.43	108.78	127.7	147.4	188.7	211.3	591.6	657	726	817
21	Sc	6.54	12.80	24.76	73.7	91.7	111.1	138.0	158.7	180.2	225.4	249.8	686	756	830
22	Ti	6.82	13.58	27.49	43.26	99.4	119.36	140.8	169.4	193.0	216.2	265.3	292	788	862
23	V	6.74	14.65	29.31	46.71	65.23	128.6	150.3	173.6	205.8	230.5	255.1	308	336	896
24	Cr	6.766	16.50	30.96	49.1	70.2	90.57	161.1	184.6	209.3	244.4	270.7	298	355	384
25	Mn	7.435	15.640	33.67	51.4	73.0	97	119.27	196.47	221.8	248.3	286.0	314	344	404
26	Fe	7.870	16.16	30.651	54.8	75.5	100	128.3	151.12	235.0	262.1	290.4	331	361	392
27	Co	7.86	17.06	33.50	51.3	79.5	103	131	160	186.2	276.2	305	336	379	411
28	Ni	7.635	18.168	35.17	54.9	75.5	108	134	164	193	224.6	321	352	384	430
29	Cu	7.726	20.292	36.83	55.2	79.9	103	139	167	199	232	266	369	401	435
30	Zn	9.394	17.964	39.72	59.4	82.6	108	136	175	203	238	274	311	412	454

TAB. 1.6 - *potentiels d'ionisation*

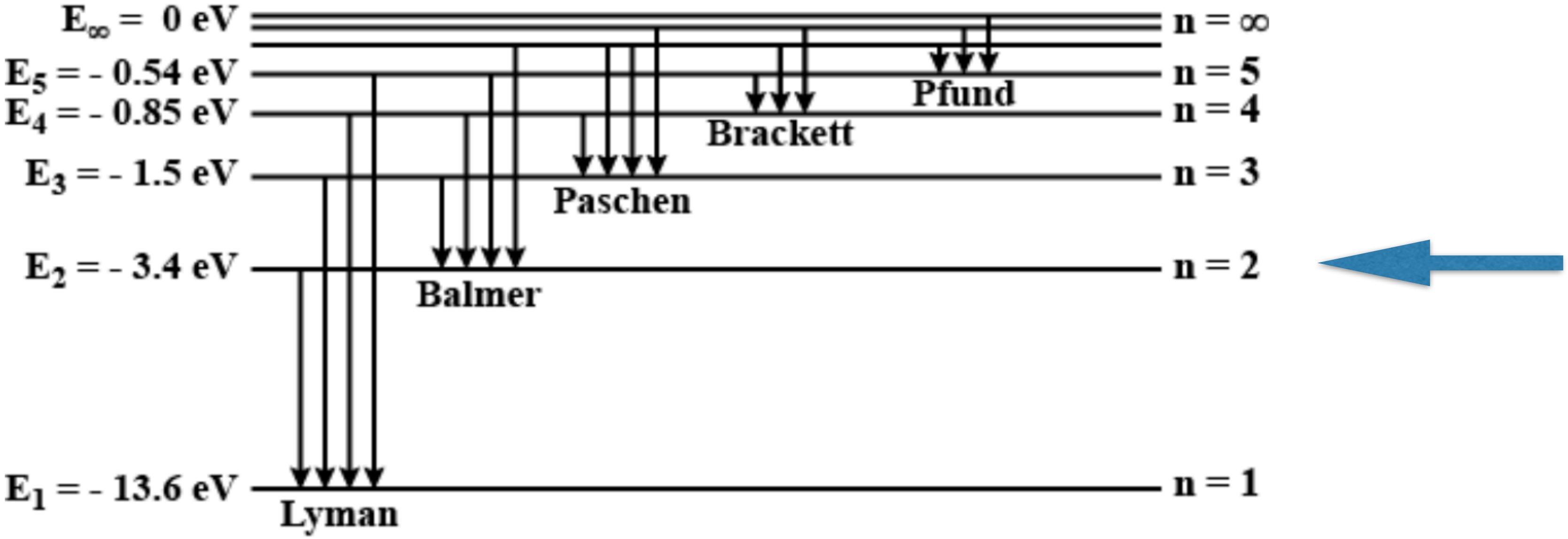


§ 15 EXCITATION, IONIZATION AND PARTITION FUNCTION 35

Partition function [1, 2, 3]

Element		$Y = \mathrm{I}$ Neutre			$Y = \mathrm{II}$ Ion I			$Y = \mathrm{III}$ Ion II
		g_0	log U		g_0	log U		g_0
			$\Theta = 1.0$	$\Theta = 0.5$		$\Theta = 1.0$	$\Theta = 0.5$	
1	H	2	0.30	0.30	1	0.00	0.00	—
2	He	1	0.00	0.00	2	0.30	0.30	1
3	Li	2	0.32	0.49	1	0.00	0.00	2
4	Be	1	0.01	0.13	2	0.30	0.30	1
5	B	6	0.78	0.78	1	0.00	0.00	2
6	C	9	0.97	1.00	6	0.78	0.78	1
7	N	4	0.61	0.66	9	0.95	0.97	6
8	O	9	0.94	0.97	4	0.60	0.61	9
9	F	6	0.75	0.77	9	0.92	0.94	4
10	Ne	1	0.00	0.00	6	0.73	0.75	9
11	Na	2	0.31	0.60	1	0.00	0.00	6
12	Mg	1	0.01	0.15	2	0.31	0.31	1
13	Al	6	0.77	0.81	1	0.00	0.01	2
14	Si	9	0.98	1.04	6	0.76	0.77	1
15	P	4	0.65	0.79	9	0.91	0.94	6
16	S	9	0.91	0.94	4	0.62	0.72	9
17	Cl	6	0.72	0.75	9	0.89	0.92	4
18	Ar	1	0.00	0.00	6	0.69	0.71	9
19	K	2	0.34	0.60	1	0.00	0.00	6
20	Ca	1	0.07	0.55	2	0.34	0.54	1
21	Sc	10	1.08	1.49	15	1.36	1.52	10
22	Ti	21	1.48	1.88	28	1.75	1.92	21
23	V	28	1.62	2.03	25	1.64	1.89	28
24	Cr	7	1.02	1.51	6	0.86	1.22	25
25	Mn	6	0.81	1.16	7	0.89	1.13	6
26	Fe	25	1.43	1.74	30	1.63	1.80	25
27	Co	28	1.52	1.76	21	1.46	1.66	28
28	Ni	21	1.47	1.60	10	1.02	1.28	21
29	Cu	2	0.36	0.58	1	0.01	0.18	10
30	Zn	1	0.00	0.03	2	0.30	0.30	1
31	Ga	6	0.73	0.77	1	0.00	0.00	2
32	Ge	9	0.91	1.01	6	0.64	0.70	1
34	Se	9	0.83	0.89	4	—	—	9
36	Kr	1	0.00	0.00	6	0.62	0.66	9
37	Rb	2	0.36	0.7	1	0.00	0.00	6
38	Sr	1	0.10	0.70	2	0.34	0.53	1
39	Y	10	1.08	1.50	1 + 15	1.18	1.41	10
40	Zr	21	1.53	1.99	28	1.66	1.91	21
48	Cd	1	0.00	0.02	2	0.30	0.30	1
50	Sn	9	0.73	0.88	6	0.52	0.61	1
56	Ba	1	0.36	0.92	2	0.62	0.85	1
57	La	10	1.41	1.85	21	1.47	1.71	10
70	Yb	1	0.02	0.21	2	0.30	0.31	—
82	Pb	9	0.26	0.54	6	0.32	0.40	1

Allen 1973



Energy level diagram for hydrogen atom

$$
\begin{aligned}
\chi &= I - hRc/n^2 \\
&= hRc - hRc/n^2 \\
&= 13.60(1 - 1/n^2)\ \text{eV}
\end{aligned}
$$

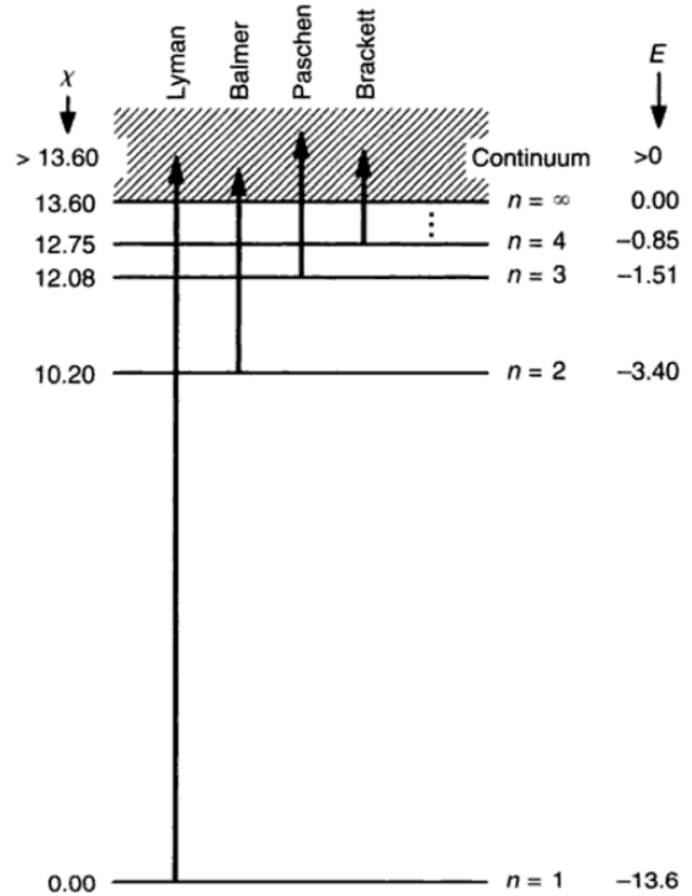
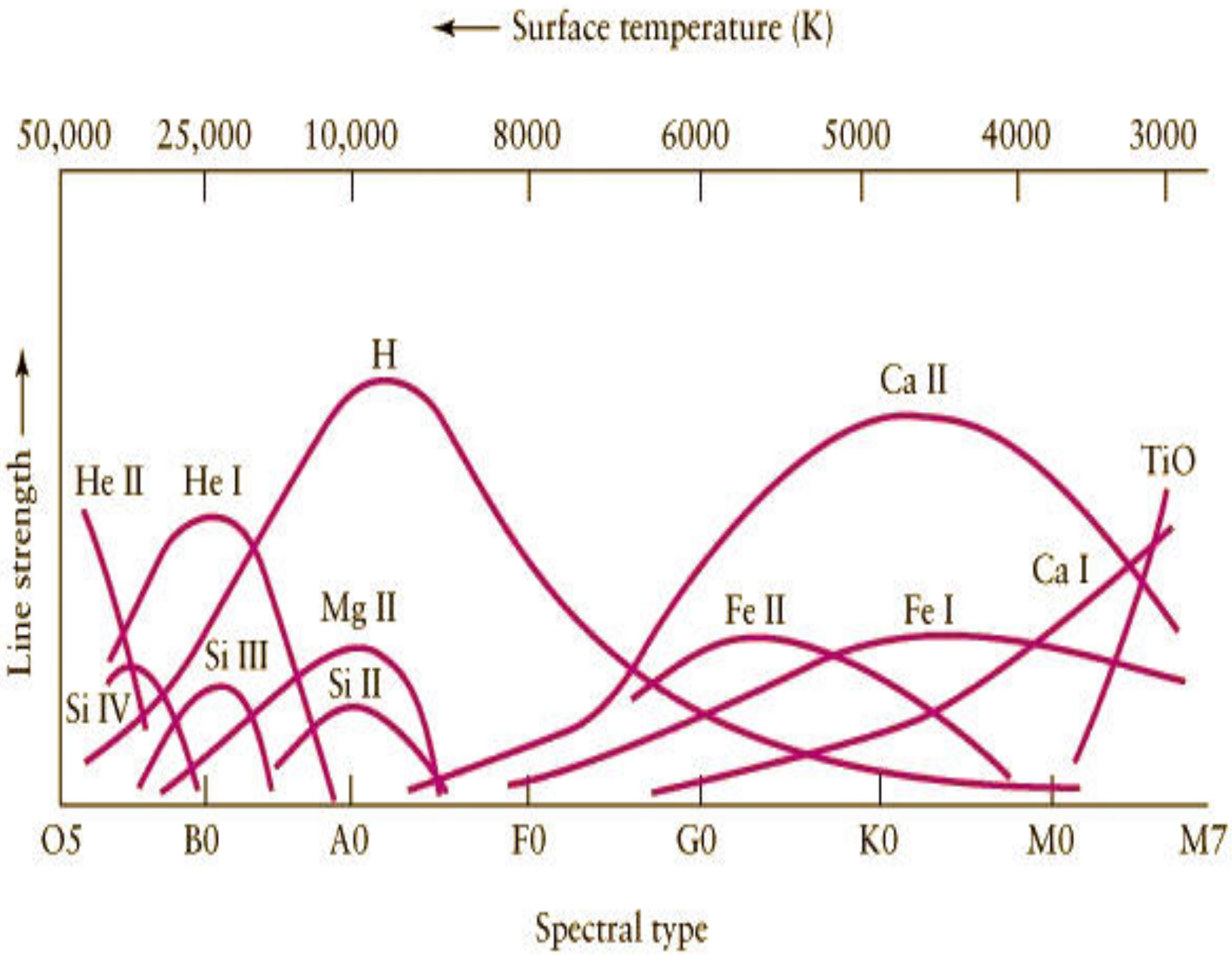


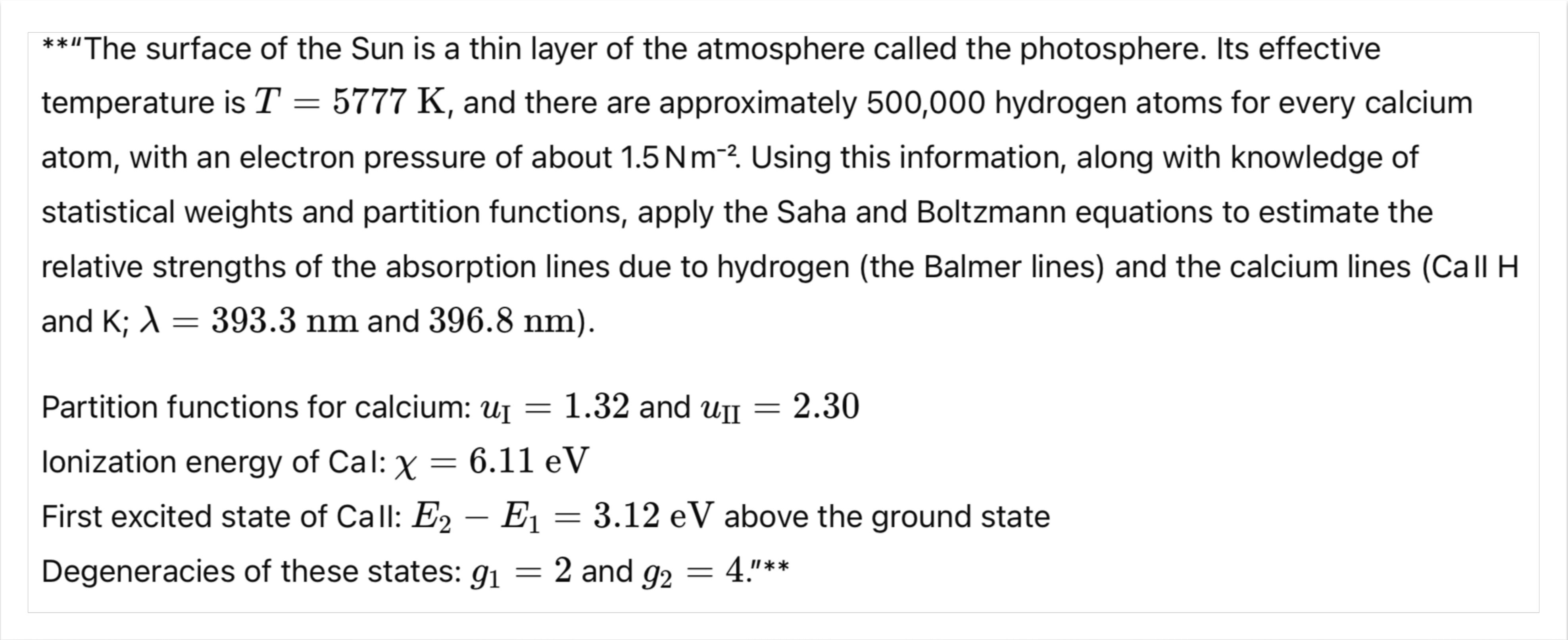
Fig. 8.1. This simplified energy-level diagram for hydrogen shows the quantum numbers, *n*, binding energy, *E*, and the excitation potential, χ, in electron volts for the first four levels and the continuum. Transitions from a bound level into the continunm comprise the bound–free absorption.

Effect of temperature in the Saha and Boltzmann equations: ionization as a function of spectral type.



At what temperature will a hydrogen gas contain equal numbers of atoms in the ground state and in the first excited state?

According to this equation, temperatures above 9520 K should lead to a higher fraction of electrons in the first excited state relative to the ground state. How, then, can we explain that the Balmer line strengths decrease at even higher temperatures?



**"The surface of the Sun is a thin layer of the atmosphere called the photosphere. Its effective temperature is $T = 5777\ \mathrm{K}$, and there are approximately 500,000 hydrogen atoms for every calcium atom, with an electron pressure of about 1.5 N m^{-2}. Using this information, along with knowledge of statistical weights and partition functions, apply the Saha and Boltzmann equations to estimate the relative strengths of the absorption lines due to hydrogen (the Balmer lines) and the calcium lines (Ca II H and K; $\lambda = 393.3\ \mathrm{nm}$ and $396.8\ \mathrm{nm}$).

Partition functions for calcium: $u_{\mathrm{I}} = 1.32$ and $u_{\mathrm{II}} = 2.30$
Ionization energy of Ca I: $\chi = 6.11\ \mathrm{eV}$
First excited state of Ca II: $E_2 - E_1 = 3.12\ \mathrm{eV}$ above the ground state
Degeneracies of these states: $g_1 = 2$ and $g_2 = 4$."**