Astrophysics II - Exercise Sheet 9

EPFL Physics section, Prof. Pascale Jablonka

1 Saha equation, partially ionized medium

Consider a medium composed of a mixture of hydrogen and helium and in which the temperature is $T = 12500$ K and the density $\rho = 10^{-7}$ kg m^{-3}. We will neglect radiation pressure. The abundances of hydrogen and helium (by number) are $\nu_1 = 0.9$ and $\nu_2 = 0.1$ respectively. Using the Saha law, we will construct a system of equations that can be solved by iteration to determine the ionization state of the medium. We first define

$$E \equiv \frac{\text{Number of free electrons}}{\text{Total number of atoms or ions}} \tag{1}$$

as well as the quantities

$$X_i^s \equiv \frac{\text{Number of atoms } i \text{ in ionization state } s}{\text{Total number of atoms or ions of species } i} \tag{2}$$

Such that we have

$$E = \sum_i \nu_i \sum_{s=0}^{Z_i} s X_i^s \tag{3}$$

In our particular case,

$$X_1^1 = \frac{\mathrm{H}^+}{\mathrm{H} + \mathrm{H}^+}, \quad X_2^1 = \frac{\mathrm{He}^+}{\mathrm{He} + \mathrm{He}^+ + \mathrm{He}^{++}}, \quad X_2^2 = \frac{\mathrm{He}^{++}}{\mathrm{He} + \mathrm{He}^+ + \mathrm{He}^{++}}, \tag{4}$$

where the fractions are intended by number. For simplicity, we will simply denote these quantities by X_1, X_2, X_3, in order.

1. Write down the equation giving E in the system considered here.

The Saha equation yields

$$\frac{X_1 \, P_e}{(1 - X_1) P_g} = K_1, \tag{5}$$

where

$$K_1 = \frac{(2\pi \, m_e)^{3/2} \, (kT)^{5/2}}{\beta \, P \, h^3} \, \omega_1 \, e^{-I_1/kT}. \tag{6}$$

P_e is the electron gas pressure, P_g is the gas pressure (including atoms, ions and electrons) and $\beta \equiv P_g/P$ is the ratio of the gas pressure to the total pressure of the medium. I_1 is the energy difference of the transition $\mathrm{H} \to \mathrm{H}^+$ and

$$\omega_1 = 2 \, \frac{U_{H^+}}{U_H}. \tag{7}$$

Furthermore, we have seen in the lecture that the gas pressure can be related to density through

$$P_g = \frac{k}{m_H} \frac{1 + E}{\mu_0} \rho T, \tag{8}$$

where μ_0 is the mean molecular weight of the atom and ion mixture, while for the electron pressure

$$P_e = \frac{k}{m_H} \frac{E}{\mu_0} \rho T. \tag{9}$$

Combining all of this and adding the Saha equations for the two other ionic species, we get the following system,

$$\left.\begin{aligned} \frac{X_1}{1-X_1}\,\frac{E}{1+E} &= K_1 \\ \frac{X_2}{1-X_2-X_3}\,\frac{E}{1+E} &= K_2 \\ \frac{X_3}{X_2}\,\frac{E}{1+E} &= K_3 \end{aligned}\right\} \tag{10}$$

which is closed by the equation for E you found in **(1)**. The K_i are defined equivalently to K_1.

2. Solve this system of equations by iteration, by starting from the initial guess that hydrogen is fully ionized and helium completely neutral. *Hint:* use this starting point to obtain an initial guess for E, and move forward from there. You can either do this "by hand", or write a short Python script to automate the process. If you choose to do the latter, try varying the temperature of the medium over the range $[5'000, 40'000]$ K and plot the fractions of He^+ and He^{++} to see how the ionization state depends on temperature.

Indications

$$\begin{aligned} I_1 &= 13.598\,eV \\ I_2 &= 24.587\,eV \\ I_3 &= 54.416\,eV \\ \omega_1 &= 2\,\frac{U_H^+}{U_H} = 2\,\cdot\,\frac{1}{2} = 1 \\ \omega_2 &= 2\,\frac{U_{He}^+}{U_{He}} = 2\,\cdot\,\frac{2}{1} = 4 \\ \omega_3 &= 2\,\frac{U_{He}^{++}}{U_{He}^+} = 2\,\cdot\,\frac{1}{2} = 1 \\ k &= 1.3807\cdot 10^{-23}\ \text{J K}^{-1} \\ h &= 6.626\cdot 10^{-34}\ \text{J S} \\ m_e &= 9.109\cdot 10^{-31}\ \text{kg} \\ m_H \simeq u &= 1.661\cdot 10^{-27}\ \text{kg} \end{aligned}$$