Astrophysics II - Exercise Sheet 1

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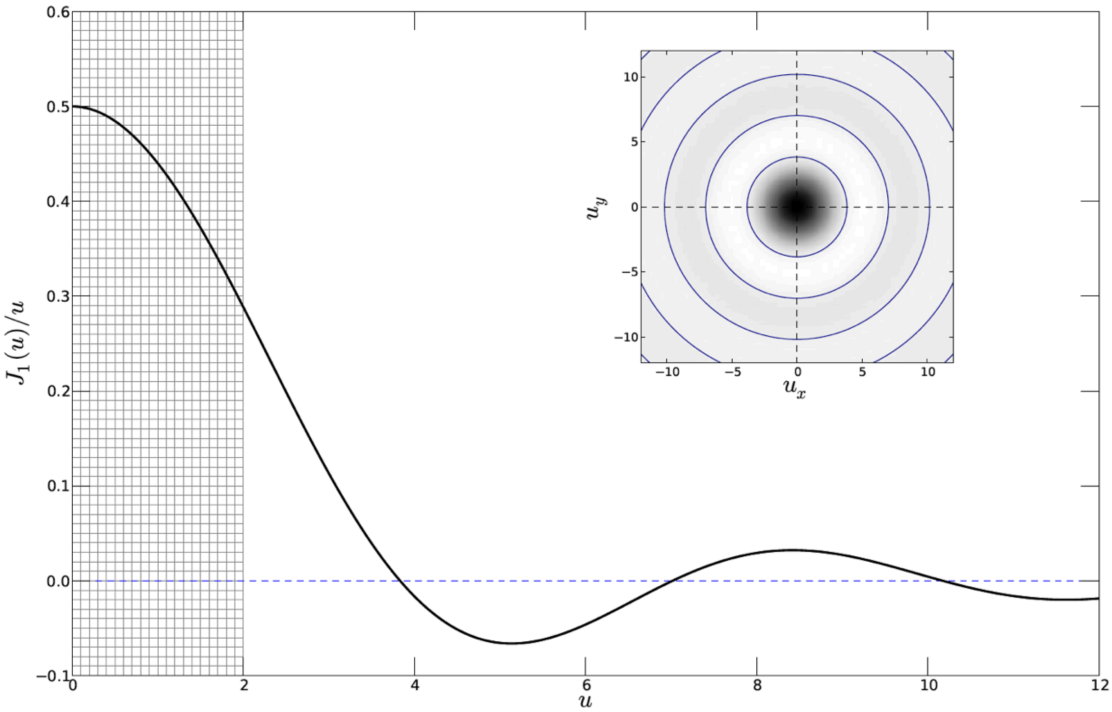
1 Resolution of a Telescope

Due to diffraction, observing a point source with a telescope of diameter D leads to a pattern of the light intensity in the focal plane that is not a Dirac distribution. Denoting by θ the angle relative to the pointing direction of the telescope, we then don't have $I(\theta) = I_0\delta(\theta)$, but instead a shape that depends on the wavelength λ,

$$I(\theta) = 4I_0 \left[\frac{J_1(u)}{u}\right]^2, \quad \text{with} \quad u = \frac{\pi D \sin\theta}{\lambda}, \tag{1}$$

where J_1 is the first-order Bessel function of the first kind. We give in the figure below the plot of $J_1(u)/u$ between 0 and 12. The inserted image represents the same function, in grayscale colours and two dimensions (u_x, u_y) with $u^2 = u_x^2 + u_y^2$. Roots of the function are shown as concentric circles.



Questions:

1. Calculate the angular diameter $\Delta\theta(\lambda, D)$ of the diffraction pattern, taken at half of the height of the central peak, as a function of the wavelength λ and telescope diameter D. This size, known as *Full-Width at Half- Maximum* (FWHM for short), is a physical limit on the angular resolution of any observational instrument in astronomy. Can you think of another effect that might limit the angular resolution ?

2. As an example, compute (numerically) $\Delta\theta$ for (a) the human eye ($D = 1$ cm, $\lambda = 0.5$ μm) (b) the Hubble Space Telescope ($D = 2.4$ m, $\lambda = 0.5$ μm) and (c) the IRAM radio-telescope at Pico Veleta ($D = 30$ m, $\lambda = 2$ mm). Express the result in arcseconds (1" $\equiv$ 1/3600°).

3. Let us consider two people standing one meter apart from each other. What is the maximum distance from which the HST could observe them while still being able to distinguish them separately ?

4. What diameter would an optical telescope such as HST need to have to be able to spatially resolve the surface of a sun-like star ($R_\odot \simeq 7 \cdot 10^8$ m) located in the Taurus molecular cloud, at a distance of 140 pc (1 pc $\simeq 3 \cdot 10^{16}$ m) ?

5. Show that in the limit $\Delta\theta << 1$, the solid angle covered by the central peak of the diffraction pattern is $\Delta\Omega \simeq \pi\Delta\theta^2/4$.

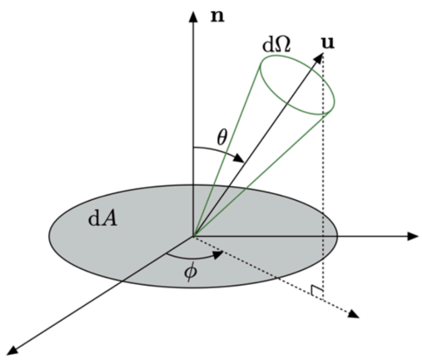
6. Deduct from this the total number of resolution elements in the whole sky for the Hubble Space Telescope observing at $\lambda = 0.5$ μm. How many pixels would a full-sky map of HST contain ? Take into account the Shannon criterion which imposes to sample the sky with an angular step at most equal to *half of the FWHM*. Estimate the corresponding data volume, assuming each pixel is encoded in one byte (8 bits).

2 Photometric Quantities

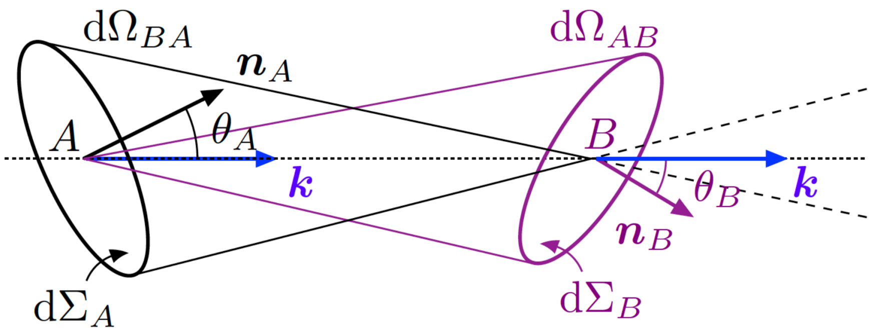
The fundamental photometric quantity in astrophysics is the *specific intensity* I_ν, also called *spectral radiance*. It is defined through the quantity of electromagnetic energy dE_ν in the frequency interval $[\nu, \nu + d\nu]$ which, in a time dt, is transported through a surface element $\vec{dA} \equiv \vec{n} dA$, into a cone of solid angle $d\Omega$ around a direction given by the unit vector $\vec{u}(\theta, \phi)$:

$$dE_\nu = I_\nu \vec{u} \cdot \vec{dA} dt d\Omega d\nu = I_\nu \cos\theta dA dt d\Omega d\nu \qquad (2)$$

We shall always assume to be in a stationary state, where I_ν is independent of time t.



We now consider the free propagation of radiation (no emission, no absorption, no diffusion). The situation is represented in the figure below, where A and B are two points along the same ray of light identified with the direction $\vec{k}$:



1. Write the conservation of radiative energy between A and B, as a function of the specific intensities $I_\nu(A,\vec{k})$ and $I_\nu(B,\vec{k})$, the angles θ_A and θ_B, the surface elements $d\Sigma_A$ and $d\Sigma_B$, the solid angle elements $d\Omega_A$ and $d\Omega_B$, and the frequency and time intervals $d\nu$ and dt. Express the solid angles $d\Omega_{A,B}$ as a function of $d\Sigma_{A,B}$, $\theta_{A,B}$ and the distance R between A and B. Conclude that the specific intensity is a conserved quantity.

2. Equivalently, one can define the specific intensity I_λ in terms of wavelength instead of frequency, such that $I_\nu|d\nu| = d_\lambda|d\lambda|$, where $d\lambda$ is the wavelength interval *corresponding to* the frequency interval $d\nu$. Calculate I_λ as a function of I_ν. Why do astronomers often use νI_ν and λI_λ instead of simply I_ν and I_λ ?

3. Next, we define the *spectral energy density* u_ν, the *spectral flux density* F_ν and the *spectral radiation pressure* p_ν,

$$u_\nu = \frac{1}{c}\int I_\nu d\Omega, \quad F_\nu = \int I_\nu \cos\theta d\Omega, \quad p_\nu = \frac{1}{c}\int I_\nu \cos^2\theta d\Omega, \tag{3}$$

where the integrals are taken over all directions (θ,ϕ). We now consider a spherical star of radius R located at a distance $D >> R$ of an observer O. Denote by θ_c the apparent angular diameter of the star seen from O, and suppose that this angle is small ($\theta_c << 1$). Show that the spectral flux density measured by O can be written as

$$F_\nu \simeq \pi I_\nu(0,0)\left[\frac{R}{D}\right]^2. \tag{4}$$

Compare this result to that of question (1) and comment.

4. Finally, we consider isotropic radiation, meaning that I_ν does not depend on θ and ϕ. Compute u_ν, F_ν and p_ν in terms of I_ν under this hypothesis. Also compute the spectral flux density F_ν^+ associated only with photons for which $\cos\theta \geq 0$.