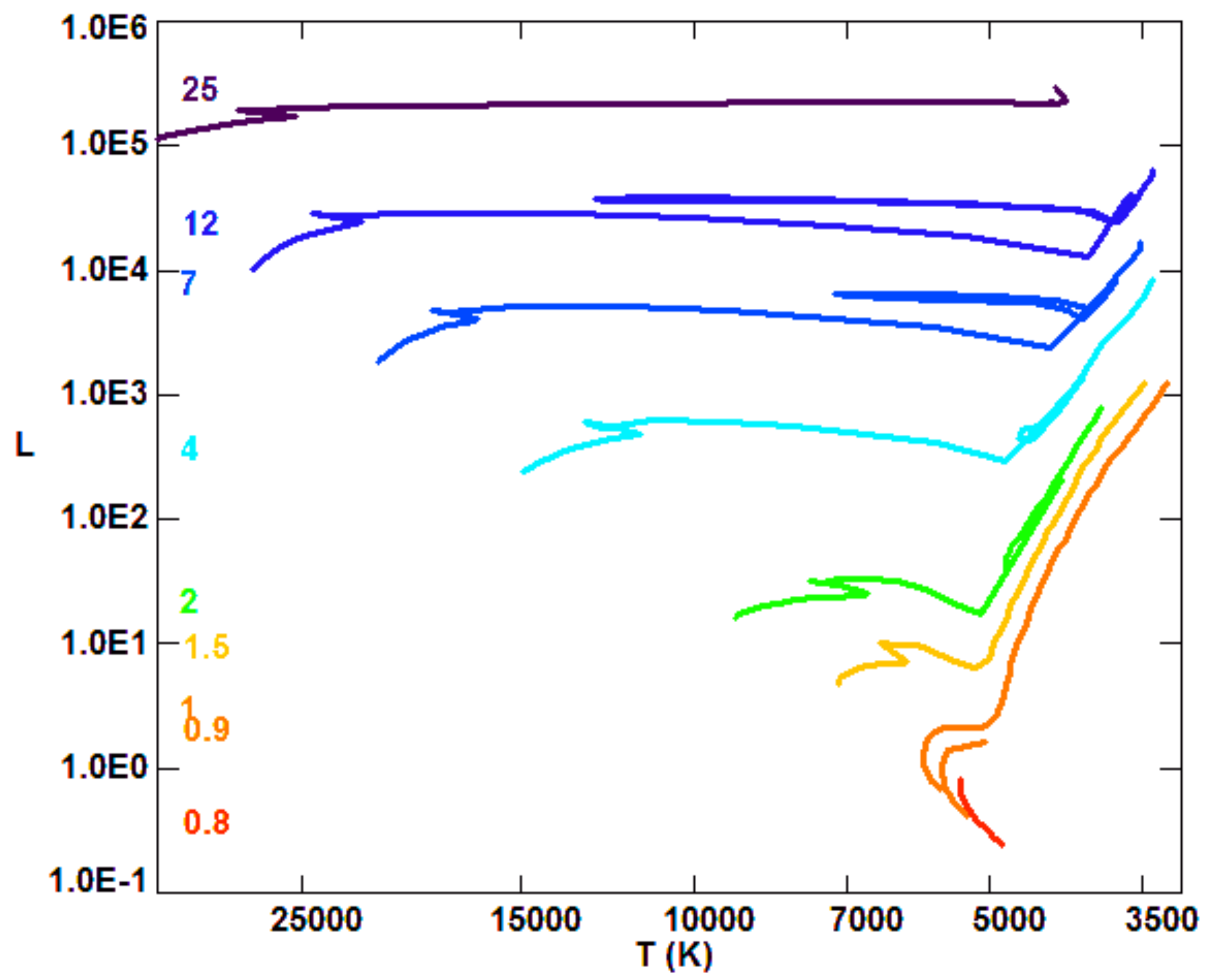
End of stellar life and degenerate gases

Degenerate gas

From the smallest to the most massive, most stars develop “degenerate” cores at some point during their evolution.



stellar evolutionary tracks in the Hertzsprung–Russell diagram as a function of stellar mass

Crédit : ASM

The Pauli Exclusion Principle was formulated by Wolfgang Pauli in 1925 while examining Bohr's atomic model. This principle states that no two identical fermions (spin-½ particles) may occupy the same quantum state simultaneously. Its application to atomic physics made it possible to understand the distribution of electrons among the atomic orbitals.

Rappels de physique quantique

n Nombre quantique principal

$0 \leq l \leq n-1$ Nombre quantique secondaire (ou azimutal)

$-l \leq m_l \leq l$ Nombre quantique tertiaire (ou magnétique

O.A. : orbitales atomiques

Valeurs de n	Valeurs de l	Nom de la sous-couche	Valeurs de m_l	Nombres d'O.A. /dégénérescence	Dégn./O.A. pour H
1	0	1s	0	1	1
2	0 1	2s 2p	0 $-1; 0; 1$	1 3	4
3	0 1 2	3s 3p 3d	0 $-1; 0; 1$ $-2; -1; 0; 1; 2$	1 3 5	9
4	0 1 2 3	4s 4p 4d 4f	0 $-1; 0; 1$ $-2; -1; 0; 1; 2$ $-3; -2; -1; 0; 1; 2; 3$	1 3 5 7	16

Hydrogène: $g_n = 2n^2$ $u(T) \sim 2$

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1	0	$1s$	0	1	1
2	0	$2s$	0	1	4
	1	$2p$	$-1; 0; 1$	3	
3	0	$3s$	0	1	9
	1	$3p$	$-1; 0; 1$	3	
	2	$3d$	$-2; -1; 0; 1; 2$	5	
4	0	$4s$	0	1	16
	1	$4p$	$-1; 0; 1$	3	
	2	$4d$	$-2; -1; 0; 1; 2$	5	
	3	$4f$	$-3; -2; -1; 0; 1; 2; 3$	7	

Hydrogène: $g_n = 2n^2$ u(T) ~ 2

Electrons are characterized by the quantum numbers n, l, m_l et m_s : For instance, if an electron has the combination (1, 0, 0, ½), no other electron in the atom can possess that exact set of quantum numbers. Consequently, each atomic shell can contain only a limited number of electrons:

This limits the number of electrons per shell:

- For $n = 1$, ($l = 0$, hence $m_l = 0$), there are only two possibilities, corresponding to the states $m_s = \pm½$.

So, only 2 electrons are possible.

- For $n = 2$, $l = 0$ ou 1 :

  ✳ for $l = 0$, $m_l = 0$;
  ✳ pour $l = 1$, $m_l = -1, 0,$ or 1 ;

There are 4 possibilities, and for each, $m_s = \pm½$, hence the second shell can accept 8 electrons (2 for $l = 0$ et 6 for $l = 1$)

In general, the n-th shell can accept $2n^2$ électrons.

So shell 3 has a maximum of 18 électrons.

The Pauli Exclusion Principle implies that ordinary matter is stable and occupies a finite volume. This idea was first articulated in 1931 by Paul Ehrenfest, who noted that the electrons of an atom cannot all collapse into the lowest-energy orbital and must instead fill increasingly larger shells. As a consequence, atoms occupy space and cannot be compressed arbitrarily closely.

In astrophysics, this principle limits the collapse of dense stellar remnants. In neutron stars, for instance, neutrons attempt to occupy the same momentum state under immense gravitational compression, but the exclusion principle generates a degeneracy pressure that opposes collapse.

However, for sufficiently massive stars, this pressure is no longer sufficient, leading to complete gravitational collapse.

End of life

There are three possible end states of stellar evolution:

- **Neutron stars**

They are the final stages of massive stars (M > 6-8 $M_\odot$).

Electrons + protons → neutrons

Neutron stars are extremely compact systems with masses comparable (factor 1.4 to 2) to the Sun (~2×10^{30} kg) and radii of ~10 km, leading to densities up to 3× to 8×10^{17} kg/m^3 [A teaspoon weighing ~4–6 billion tons.].

- **Black holes**

If the core is greater than about 2-3 solar masses (the maximum mass of a neutron star), the pressure of neutrons is unable to stop the collapse and a stellar black hole is formed.

http://ligo.org/

https://pnp.ligo.org/ppcomm/Papers.html

GW190521: A Binary Black Hole Merger with a Total Mass of 150 Msun (*by LSC and Virgo*)
GW231123: A Binary Black Hole Merger with Total Mass 190-265 M_sun (*by LSC, Virgo and KAGRA*
GW190814: Gravitational Waves from the Coalescence of a 23 Solar Mass Black Hole with a 2.6 Solar Mass Compact Object (*by LSC and Virgo*)
Observation of Gravitational Waves from a Binary Black Hole Merger (*by B. P. Abbott et al. (LSC and Virgo)*)

LIGO has two long arms at right angles (4 km each).
A laser beam is split into two beams.
Each beam travels down one arm and bounces between mirrors many times (effective length ≈ 1000 km).
The beams return to the beam splitter and interfere.
If the arms have exactly the same length, the recombined light cancels out (dark fringe).

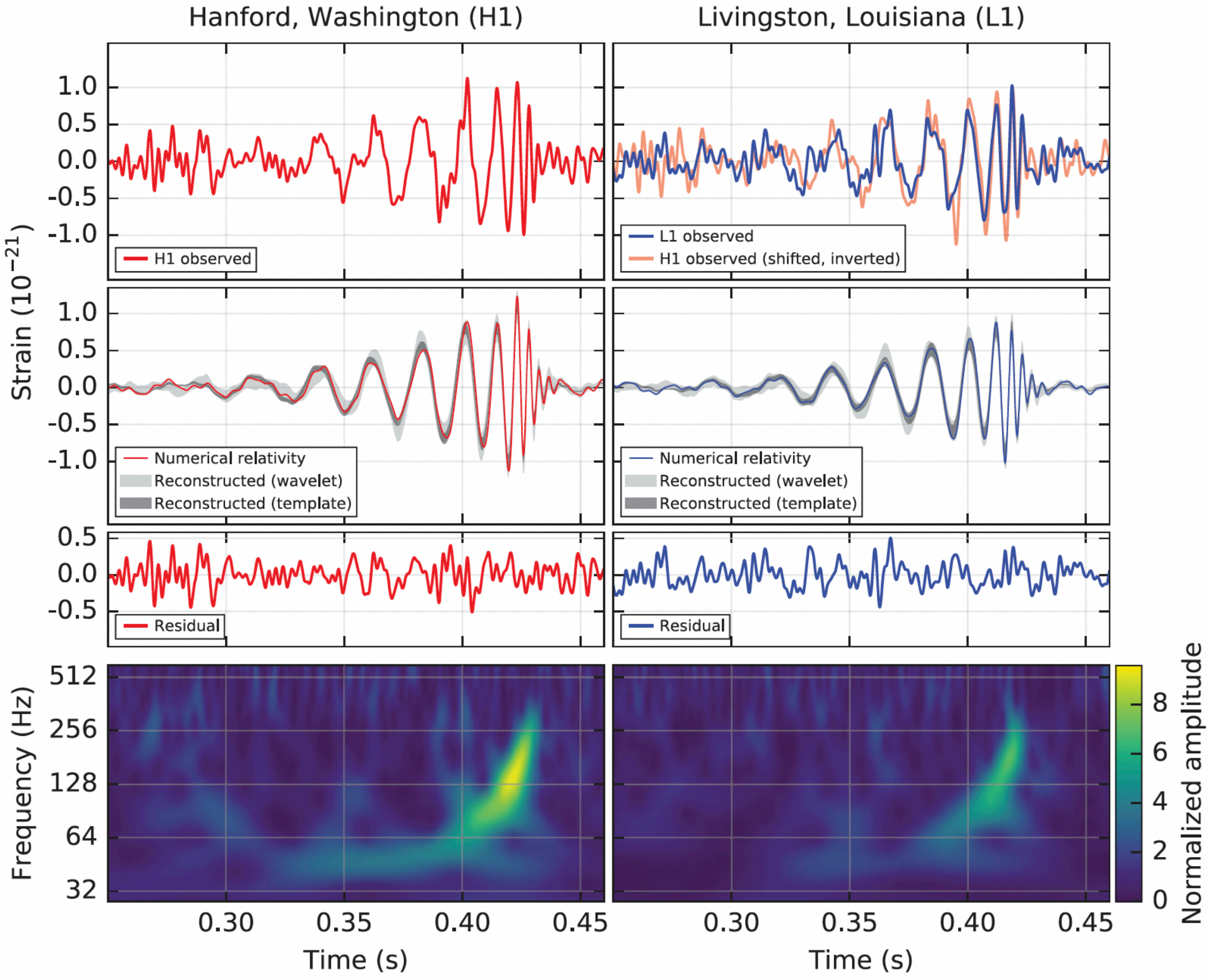


FIG. 1. The gravitational-wave event GW150914 observed by the LIGO Hanford (H1, left column panels) and Livingston (L1, right column panels) detectors. Times are shown relative to September 14, 2015 at 09:50:45 UTC. For visualization, all time series are filtered with a 35–350 Hz bandpass filter to suppress large fluctuations outside the detectors' most sensitive frequency band, and band-reject filters to remove the strong instrumental spectral lines seen in the Fig. 3 spectra. *Top row, left:* H1 strain. *Top row, right:* L1 strain. GW150914 arrived first at L1 and $6.9^{+0.5}_{-0.4}$ ms later at H1; for a visual comparison, the H1 data are also shown, shifted in time by this amount and inverted (to account for the detectors' relative orientations). *Second row:* Gravitational-wave strain projected onto each detector in the 35–350 Hz band. Solid lines show a numerical relativity waveform for a system with parameters consistent with those recovered from GW150914 [37,38] confirmed to 99.9% by an independent calculation based on [15]. Shaded areas show 90% credible regions for two independent waveform reconstructions. One (dark gray) models the signal using binary black hole template waveforms [39]. The other (light gray) does not use an astrophysical model, but instead calculates the strain signal as a linear combination of sine-Gaussian wavelets [40,41]. These reconstructions have a 94% overlap, as shown in [39]. *Third row:* Residuals after subtracting the filtered numerical relativity waveform from the filtered detector time series. *Bottom row:* A time-frequency representation [42] of the strain data, showing the signal frequency increasing over time.

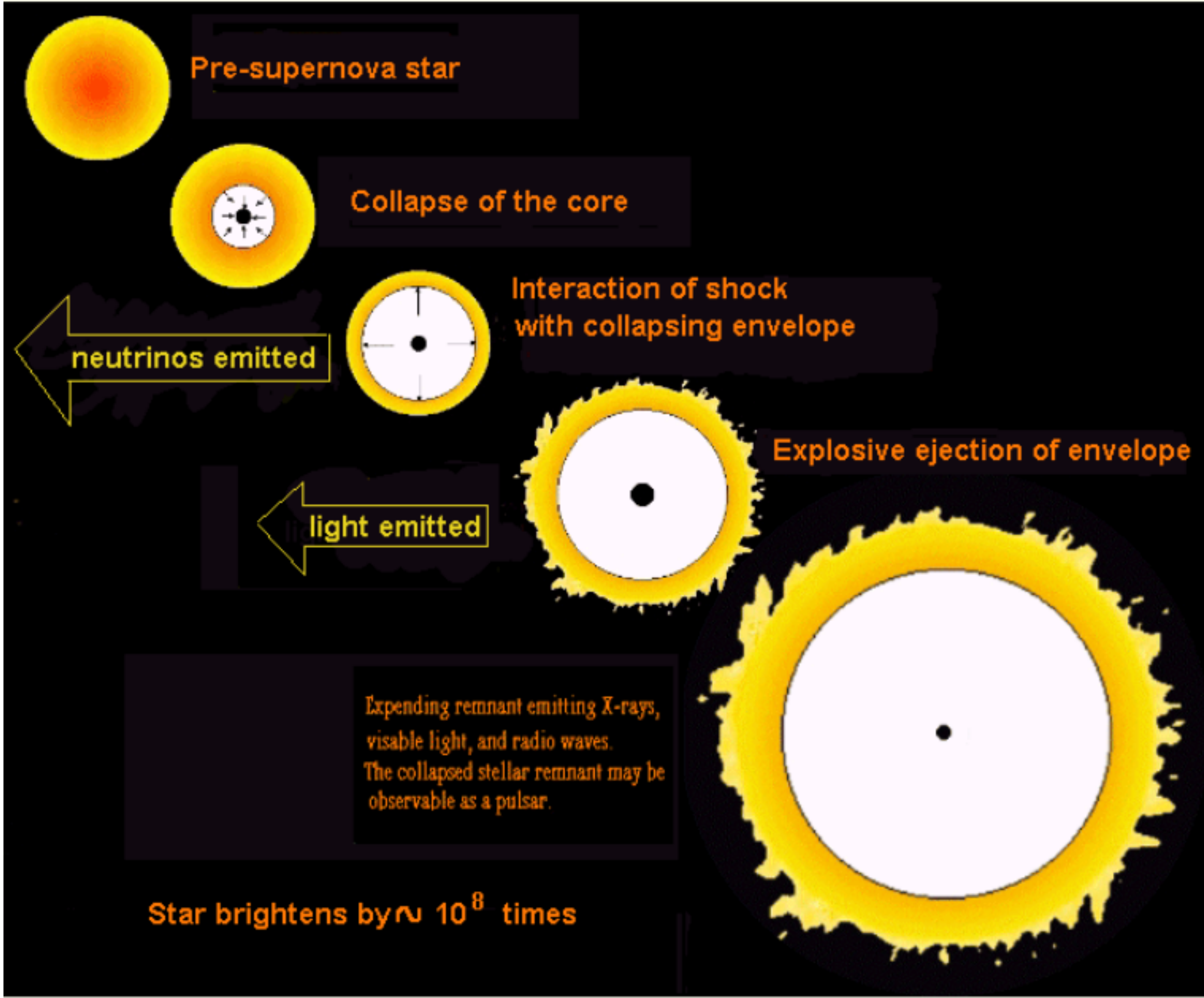
Massive-star evolution to neutron stars

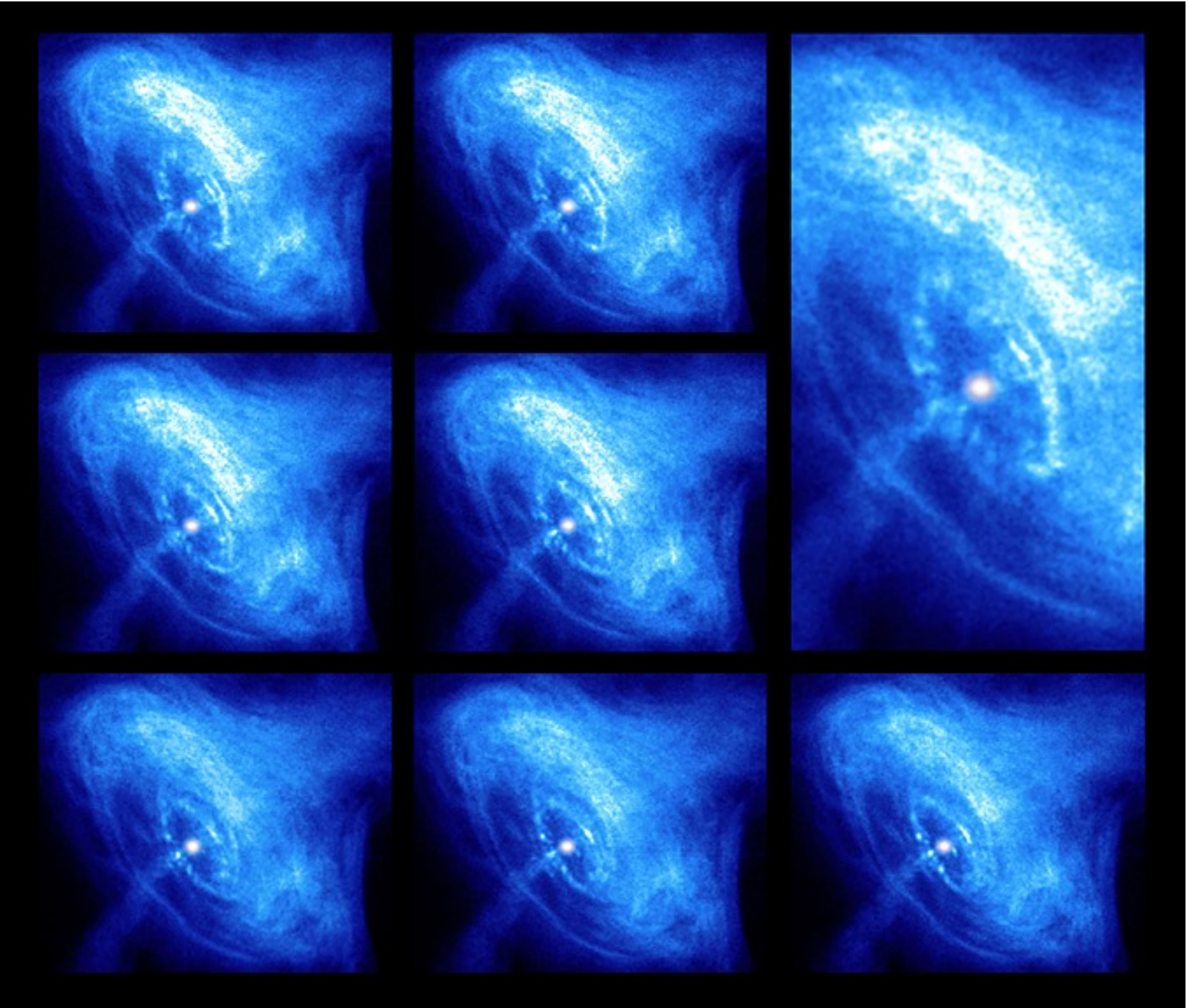
Massive stars, typically greater than eight times the mass of our sun are energetic enough to fuse elements up to iron. Fusion can no longer naturally occur in stars past iron, however, because it becomes an endothermic process, thus requiring energy. When iron is formed, it is deposited in the core, causing the density to rapidly increase and the core to begin to contract inward. This causes temperatures to rise in the core to help resist collapse. The rise in temperature and density allows for electron capture in the core by the reaction,

$p^+ + e^- \leftrightarrow n + \nu_e$ (1)

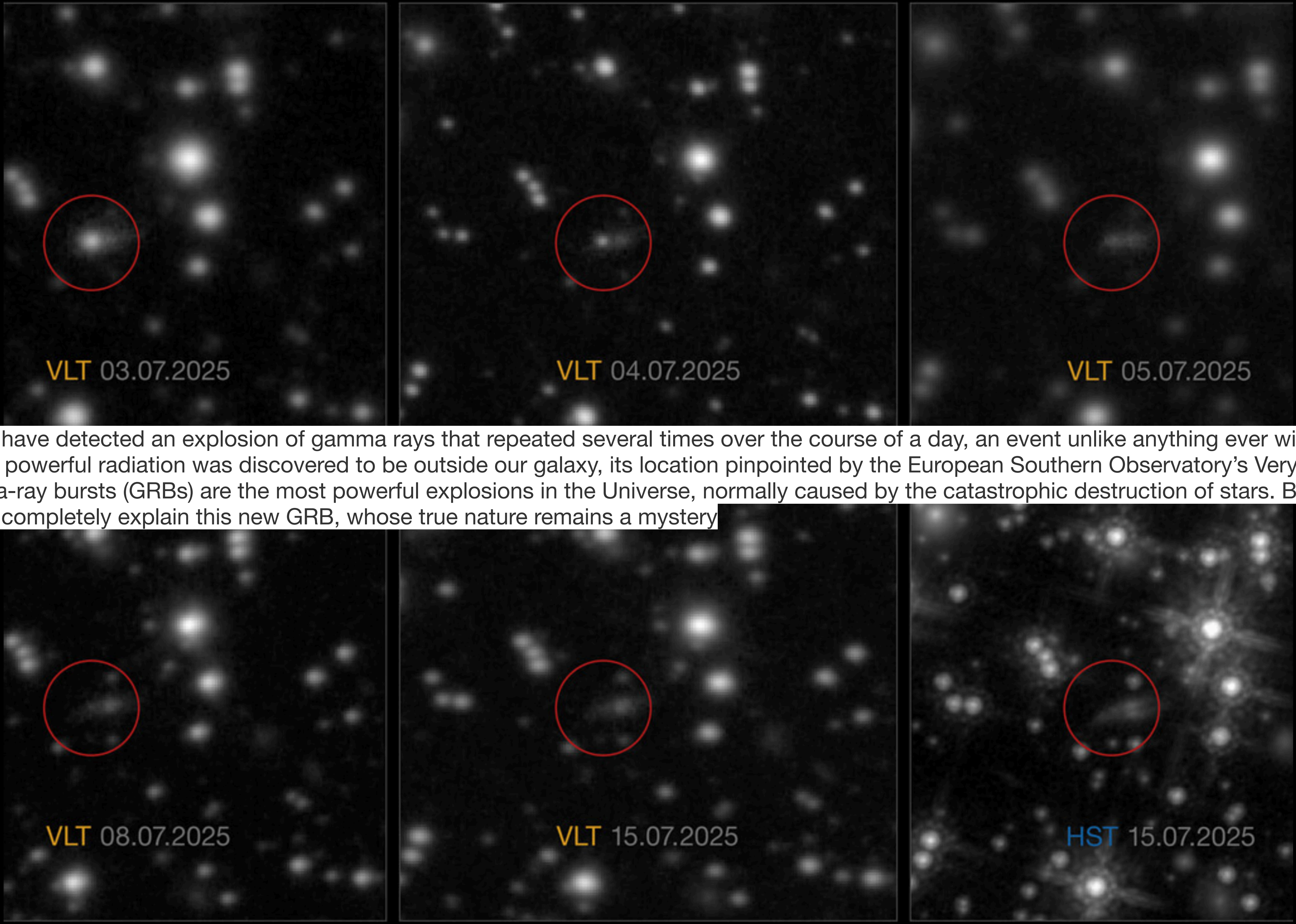
Both neutrinos and neutron rich matter are produced at the core of these large stars. Eventually, the core of these larger stars will become too massive, causing a gravitational core collapse supernova which, in many cases, leave behind a neutron star. These neutron stars are neutron rich due to reaction (1), and can weigh up to three solar masses. Neutron stars are much denser than white dwarf stars, which, once again, causes the core of the stars to collapse. The compression of neutrons in the contracting core, however, creates a neutron degeneracy pressure. This pressure, analogous to the electron degeneracy pressure in white dwarf stars, combats the gravitational collapse of the star. If, however, the neutron star is too massive (more than three solar masses), the neutron degeneracy pressure fails and the neutron star collapses into a black hole.

- Within about 0.1 second, the core collapses.
- After about 0.5 second, the collapsing envelope interacts with the outward shock. Neutrinos are emitted.
- Within 2 hours, the envelope of the star is explosively ejected. When the photons reach the surface of the star, it brightens by a factor of 100 million.
- Over a period of months, the expanding remnant emits X-rays, visible light and radio waves in a decreasing fashion.





The Chandra images in this collage were made over a span of several months (ordered left to right, except for the close-up). They provide a stunning view of the activity in the inner region around the Crab Nebula pulsar, a rapidly rotating neutron star seen as a bright white dot near the center of the images.
A wisp can be seen moving outward at half the speed of light from the upper right of the inner ring around the pulsar.
The wisp appears to merge with a larger outer ring that is visible in both X-ray and optical images.
The inner X-ray ring consists of about two dozen knots that form, brighten and fade.



Astronomers have detected an explosion of gamma rays that repeated several times over the course of a day, an event unlike anything ever witnessed before. The source of the powerful radiation was discovered to be outside our galaxy, its location pinpointed by the European Southern Observatory's Very Large Telescope (VLT). Gamma-ray bursts (GRBs) are the most powerful explosions in the Universe, normally caused by the catastrophic destruction of stars. But no known scenario can completely explain this new GRB, whose true nature remains a mystery

- **White dwarfs**

White dwarfs constitute the final evolutionary stage of ~90% of stars. They are compact objects with densities of order (10^6g/cm^3) and radii comparable to that of Earth (~6,371 km)

White dwarfs are compact, electron–degenerate remnant cores of low- or intermediate-mass stars (initial mass ≲ 8–10 M⊙) that have terminated nuclear fusion. After such a star evolves off the main sequence, passes through the red-giant and asymptotic giant-branch (AGB) phases, and expels its outer envelope as a planetary nebula, the inert core is left behind as a white dwarf.

The majority of white dwarfs have **carbon–oxygen (C/O)** cores, originating from stars with initial masses of ~1–8 M⊙. More massive progenitors may form **oxygen–neon–magnesium (ONeMg)** white dwarfs, while very low-mass white dwarfs (≲ 0.45 M⊙) are often **helium white dwarfs**, typically produced by binary evolution that strips the envelope before helium ignition.

White dwarfs no longer sustain nuclear burning. Their luminosity derives from residual thermal energy, and they cool monotonically over gigayear timescales following a well-defined mass–radius and luminosity–temperature relation. Typical radii (~0.01 R⊙) are comparable to Earth's, while their masses cluster around ~0.6 M⊙. Their cooling sequences and spectral characteristics (e.g., DA with hydrogen atmospheres, DB with helium atmospheres) are central tools for age-dating stellar populations.

Transition from non-degenerate to degenerate cores as nuclear reactions cease

On the MS a star is chemically homogenous, at least at the very beginning, and derives its energy from the nuclear burning of hydrogen to helium at the centre. The part of the MS shown in the figure refers to the range of stellar masses between 0.7 Msun (lower end) and 3.5 Msun (high end). **The evolutionary track shown is calculated for 1 *Msun*.** Due to the high energy gained from the transformation of hydrogen to helium and the high abundance of hydrogen (90% by numbers), this is a very long-lived phase and takes of the order of 8 **x** 10^9 yr for a solar mass star. During this phase the luminosity increases slowly.

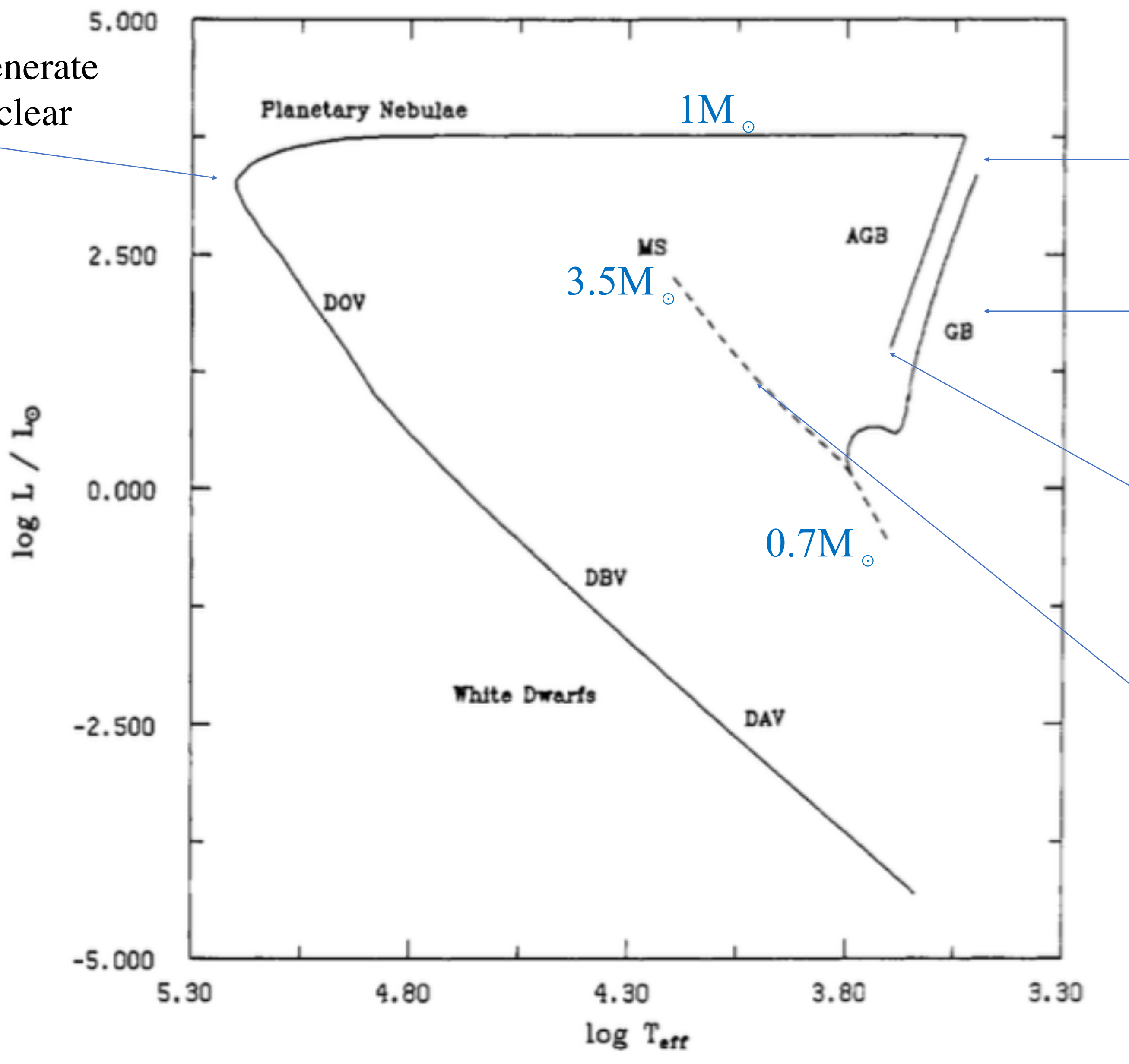


Figure 4. Qualitative description of the prehistory of a typical white dwarf.

Towards C and O
Partially degenerate core
Helium flash

He, expansion & cooling

Equilibrium, non degenerated conditions

main-sequence timescales ~ 8 10^9yr

~90% of the star composition = H

D Koester and G Chanmugam

Formation of degenerate electron gas in white dwarfs

First white dwarf discovered: the companion of Sirius (1862).

The contraction of a solar-type star compresses the electrons in its core into highly degenerate energy levels. The resulting electron degeneracy pressure (Fermi pressure) counterbalances gravity, stabilizing the white dwarf.

Most white dwarfs originate from red giants that have exhausted their nuclear fuel and lost a large fraction of their mass. These stars have fused hydrogen and helium, producing carbon and frequently oxygen in their cores. Once nuclear fusion cannot progress beyond carbon, the star contracts; the electrons become degenerate, forming Fermi pressure that supports the white dwarf against gravity.

Because the maximum mass of a neutron star is about 1.4 $M_\odot$ (the Chandrasekhar mass), the difference between the initial mass and this value must be lost into the interstellar medium.

Beyond their importance for the chemical evolution of galaxies (the study of metals dispersed into the interstellar medium, reused for subsequent generations of stars, and the metals locked inside white dwarf cores), white dwarfs are also interesting in their own right. At these extreme densities, electrons are degenerate, and their nature—described by quantum mechanics through the Pauli principle—determines their equation of state, their structure, and the existence of a limiting mass.

The Pauli Exclusion Principle, was formulated by Wolfgang Pauli in 1925 while he was considering Bohr's atomic model, and in order to attempt to explain the results of his experiments on the Zeeman Effect in atomic spectroscopy. This principle states that no two identical fermions (spin 1⁄2 particles) can occupy the same quantum state at the same time. The application of this principle to atoms, in the case of Neils Bohr, has allowed physicists to better explain how electrons fill orbitals around a nucleus of an atom, satisfying the different selection rules that atom.

We now see that the role of both the neutron degeneracy pressure, and the electron degeneracy pressure is crucial to the maintained stability of a star. We must now ask what neutron or electron degeneracy is, and how it forms a pressure in a star.

Role of degeneracy pressure (from neutrons/electrons) in maintaining stellar stability

Pauli exclusion principle for half-integer–spin fermions:

Heisenberg: $\Delta^3 p_i \, \Delta^3 q_i \geq h^3$ meaning $\Delta q_i \searrow$ (confinement) then $\Delta p_i \nearrow$

and the kinetic energy gained is a source of non-thermal pressure.

We consider an ionized medium

electrons	m_e	p_e
nuclei	m_N	p_N

The total energy of a system in thermodynamic equilibrium is, on average, equally distributed among its various components (energy equipartition).

$$\frac{p_N^2}{2m_N} = \frac{p_e^2}{2m_e},$$

$$\frac{p_N}{p_e} = \left(\frac{m_N}{m_e}\right)^{1/2} = A^{1/2} \left(\frac{m_H}{m_e}\right)^{1/2} = 43\, A^{1/2} >> 1$$

Which means that in three dimensions the momentum domain of the nuclei is larger than that of the electrons as

$$A^{3/2}(m_H/m_e)^{3/2}$$

Electrons are degenerate at densities $\rho \sim 10^4$ g/cm^3

Neutrons are fully degenerate at densities $\rho \sim 10^{14}$ g/cm^3

A gas becomes degenerate when the thermal energy kT. is much smaller than the Fermi energy E_F

$$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} \qquad \text{n, number density} \qquad p_F = \hbar(3\pi^2 n)^{1/3}$$

In **astrophysics,** temperatures are far below the Fermi temperature, so **density alone determines degeneracy**.

- White dwarfs: $T \sim 10^5$–10^7 K
- Neutrons stars: $T \sim 10^8$–10^9 K

The **Fermi temperature** of degenerate matter is :

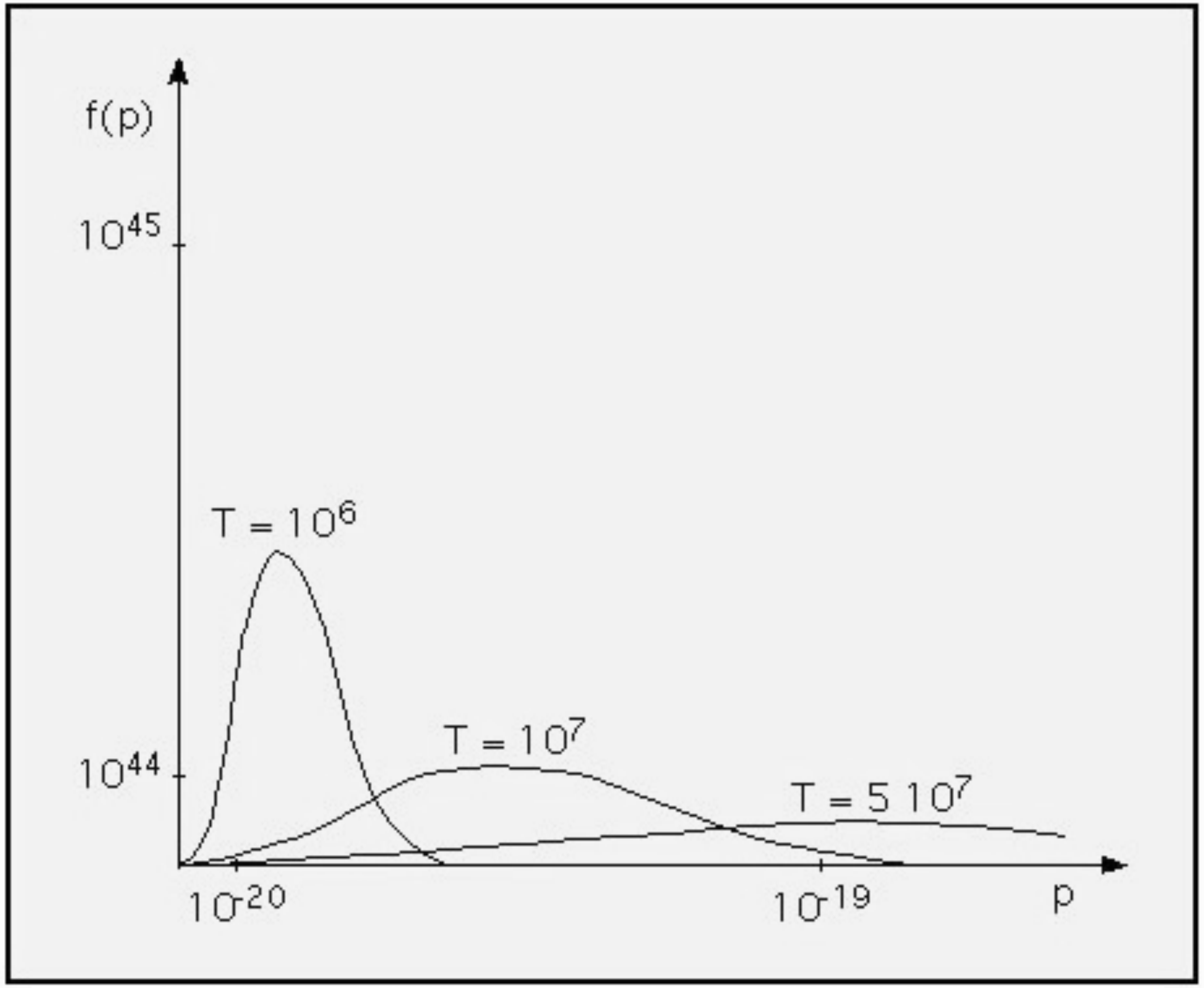
- Electrons in white dwarfs: $T_F \sim 10^9$–10^{10} K
- Neutrons in neutron stars: $T_F \sim 10^{12}$ K

In classical mechanics:

At absolute zero, all electrons have the same velocity—namely, zero.
If the temperature is increased slightly, the electrons begin to move, but they cannot move rapidly, since their average velocity reflects the temperature. Thus, the electrons share the small amount of available energy and all acquire essentially the same velocity (and therefore the same momentum).

As the temperature increases further, the **average** velocity of the electrons increases. However, the range of possible velocities between zero and this average velocity becomes larger. Consequently, the electrons distribute themselves over this expanding range. When the temperature becomes very high, their distribution becomes nearly uniform,

$f(p)$, representing the number of electrons as a function of velocity (or momentum), exhibits a pronounced peak at low temperature, and this peak becomes increasingly broad and diffuse as the temperature rises.

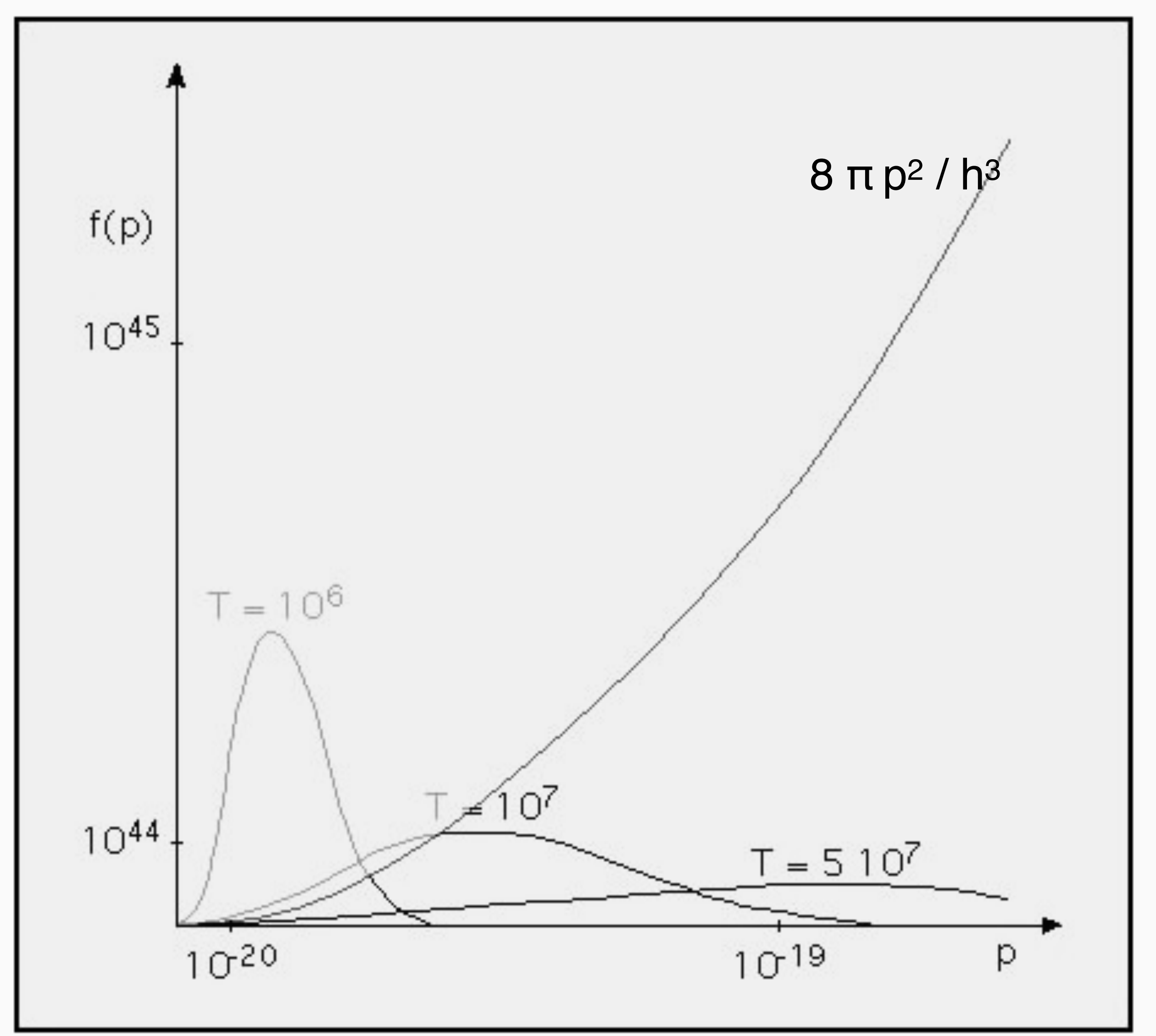


Number f(p) of electrons as a function of momentum

For example, at 10^6 K, the number of electrons with a momentum of 10^{-20} kg m/s is very large.

If matter is highly compressed, subject to a high density, the electrons will be very close to each other. If they have identical momenta (identical velocities), they will be at the same place and have the same velocity at the same time, which is forbidden by the Pauli exclusion principle

Given a position and a momentum, there can be only two electrons, with opposite spins. This therefore limits the number of electrons having a given momentum. It is easy to show that this number must be less than $8 \pi p^2 / h^3$



The Pauli principle therefore requires that the number of electrons be smaller than the value given by the parabola, that is, below it.

We thus see that everything behaves properly at high temperature ($5\ 10^7$ K), But at a much lower temperature, the exclusion principle is violated by this classical approach.

All of this has been calculated at constant density. If the density is increased, the electrons move closer together, and the Pauli principle is violated even more severely.

The degree of degeneracy increases if:

– the temperature decreases;

– the density increases.

Given N fermions, with total energy E, N_r fermions with energy E_r

$$E = \sum_r N_r\, E_r \qquad N = \sum_r N_r \qquad N_r = \frac{g_r}{e^{-\psi + \frac{E_r}{kT}} + 1},$$

g_r is the statistical weight of the particles of energy E_r

For electrons, which have two spin states, $$g_r = \frac{2}{h^3}\,\Delta^3 p_r\,\Delta^3 q_r$$

When transitioning from a discrete distribution to a continuous and isotropic distribution, the particle concentration in momentum space can be written as:

$$n(p)d^3p = \frac{8\pi p^2 dp}{h^3}\,\underbrace{\frac{1}{e^{-\psi + \frac{E}{kT}} + 1}}_{q\,(\psi, E/kT)}$$

number of particles per unit volume

q = occupation probability of phase-space cells

In a system composed of fermions, each fermion occupies a different quantum state defined by its quantum numbers, with these states being filled in order of increasing energy:

the lowest-energy state is occupied first,
then, depending on the number of fermions, states of increasingly higher energy are occupied.

The energy of the highest occupied state is called the Fermi energy.

- A gas is non-degenerate if $q << 1$ hence for $n(p)d^3p << 8\pi p^2dp/h^3$

and $\psi \ll -1$

$$q \simeq e^{\psi} \cdot e^{-E/kT}$$

The distribution is given by the Maxwell–Boltzmann law

- A gas is fully generate when $\psi \rightarrow \infty$

If energy $E < E_F$	q=1	all states are occupied
If energy $E > E_F$	q=0	all states are empty

Note that if all the cells in phase space are occupied by two particles $n(p)d^3p = \frac{8\pi p^2 dp}{h^3}$

$q = \frac{1}{2}$ for $\frac{E}{kT} = \psi$

Even for $T = 0$, the particles have non zero velocities.

For increasing densities, we have the following sequence (at constant temperature):

Ideal gas,

- partially degenerate
- completely degenerate, non-relativistic
- completely degenerate, partially relativistic
- completely degenerate, relativistic

We will now look for the expressions of pressure and internal energy, and then we will examine the two limiting cases.

$\psi << -1$: weak degeneracy

$\psi >> +1$: strong degeneracy

Electronic concentration $$n_e = \iiint n(p)\, d^3p = \frac{8\pi}{h^3} \int_0^{\infty} \frac{p^2\, dp}{e^{-\psi + \frac{E}{kT}} + 1}$$

Density $$\rho = \mu_e\, m_H\, n_e$$

Pressure $$P = \frac{1}{3} \iiint n(p)\, p\, v\, d^3 p$$ (Cf initial lecture chapter(s))

$$P = \frac{8\pi}{3\, h^3} \int_0^{\infty} v\, \frac{p^3\, dp}{e^{-\psi + \frac{E}{kT}} + 1}$$

Given that $n(p)\, d^3\, p$ is the spatial concentration

$$u = \int_0^{\infty} E\, n(p)\, d^3\, p = \frac{8\pi}{h^3} \int_0^{\infty} \frac{E\, p^2\, dp}{e^{-\psi + \frac{E}{kT}} + 1},$$

Partially degenerate gas

$$E = \frac{p^2}{2\,m_e}.\ \text{Let } x = \frac{E}{kT} = \frac{p^2}{2m_e kT}.$$

$$n_e = \frac{8\pi}{h^3}\frac{(2\,m_e\,kT)^{3/2}}{2}\int_0^\infty \frac{x^{1/2}dx}{e^{-\psi+x}+1} = \frac{4\pi}{h^3}(2\,m_e\,kT)^{3/2}F_{1/2}(\psi)$$

Fermi integral $F_n(\psi) = \int_0^\infty \frac{x^n\,dx}{e^{-\psi+x}+1}$ tabulated for $F_{1/2},\ F_{3/2}$

With $v = \frac{p}{m_e}$

The pressure is $P = \frac{8\pi}{3\,h^3\,m_e}\int_0^\infty \frac{p^4\,dp}{e^{-\psi+E/kT}+1} = \frac{8\pi}{3\,h^3}kT\,(2\,m_e\,kT)^{3/2}\underbrace{\int_0^\infty \frac{x^{3/2}\,dx}{e^{-\psi+x}+1}}_{F_{3/2}(\psi)}$

The equation of state is defined by two equations

$$P = \frac{8\pi}{3\,h^3}(2\,m_e\,kT)^{3/2}\,kT\,F_{3/2}(\psi)$$

$$\rho = \frac{4\pi}{h^3}(2\,m_e\,kT)^{3/2}\,\mu_e\,m_H\,F_{1/2}(\psi)$$

Case of very weak degeneracy ($\psi \ll -1$)

$$F_n(\psi) \longrightarrow e^{\psi} \int_0^{\infty} e^{-x} x^n \, dx$$

$$\int_0^{\infty} x^n \, e^{-ax} \, dx = \frac{\Gamma(n+1)}{a^{n+1}}$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$\Gamma(1.5) = \frac{\sqrt{\pi}}{2} \quad \text{so} \quad F_{1/2}(\psi) = \frac{\sqrt{\pi}\, e^{\psi}}{2}$$

$$\Gamma(2.5) = \frac{3}{4}\sqrt{\pi} \qquad F_{3/2}(\psi) = \frac{3}{4}\sqrt{\pi}\, e^{\psi}$$

$$P = \frac{8\pi}{3\,h^3}\,(2\,m_e\,kT)^{3/2}\;kT\,\frac{3}{4}\,\sqrt{\pi}\,e^{\psi}$$

$$\rho = \frac{4\pi}{h^3}\,(2\,m_e\,kT)^{3/2}\;\mu_e\,m_H\,\frac{\sqrt{\pi}}{2}\,e^{\psi}$$

$$\frac{P}{\rho} = \frac{kT}{\mu_e\,m_H}$$

Law of ideal gas

Case of very strong degeneracy ($\psi \gg 1$)

$$F_n(\psi) = \int_0^{\psi} x^n \, dx = \frac{\psi^{n+1}}{n+1}$$

$$x = \frac{E}{kT}$$

$$F_n = \frac{1}{n+1}\left(\frac{E_F}{kT}\right)^{n+1}$$

Given that

$$P = \frac{8\pi}{3\,h^3}\,(2\,m_e\,kT)^{3/2}\,kT\,F_{3/2}(\psi)$$

$$\rho = \frac{4\pi}{h^3}\,(2\,m_e\,kT)^{3/2}\,\mu_e\,m_H\,F_{1/2}\,(\psi)$$

replacing $F_{1/2},\ F_{3/2}$ by their expressions

We get

$$P = K_1\left(\frac{\rho}{\mu_e}\right)^{5/3}$$

Full degeneracy (non relativistic) $\psi \to \infty$

$q\,(\psi,\, E/kT) = 1$ up to p_F and zero above

$$n_e = \frac{8\pi}{h^3} \int_0^{p_F} p^2\, dp = \frac{8\pi}{3\, h^3}\, p_F^3$$

$$\frac{\rho}{\mu_e} = \frac{8\pi}{3}\, \frac{m_H}{h^3}\, p_F^3$$

We take this expresion to get p_F at given ρ

$$P = \frac{8\pi}{3\, h^3} \int_0^{p_F} \frac{p}{m_e}\, p^3\, dp = \frac{8\pi}{15\, h^3}\, \frac{p_F^5}{m_e}$$

removing p_F,

$$P = K_1 \left(\frac{\rho}{\mu_e} \right)^{5/3}$$

$$K_1 = \frac{8\pi}{15\, h^3\, m_e} \left(\frac{3\, h^3}{8\pi\, m_H} \right)^{5/3} = 1.004 \cdot 10^{13}\ \mathrm{CGS}$$

Full degeneracy (relativistic)

$v \to c.\ n_e$ and ρ/μ_e are given as in the NR case

$$P = \frac{8\pi}{3\,h^3}\int_0^{p_F} c\,p^3\,dp = \frac{2\pi\,c}{3\,h^3}\,p_F^4$$

$$P = K_2\left(\frac{\rho}{\mu_e}\right)^{4/3} \qquad K_2 = \frac{hc}{8\,m_H}\left(\frac{3}{\pi\,m_H}\right)^{1/3}$$

In the relativistic case, a 1% increase in density produces a 4/3% increase in pressure, whereas in the non-relativistic case, it produces a 5/3% increase in pressure.

Why is the relative increase in pressure smaller in the relativistic case? Physically, this results from the fact that when velocities are much less than c , an increase in the energy of the electrons translates almost exclusively into an increase in velocity, and therefore into an increase in pressure. In the relativistic case, an increase in energy also implies an increase in inertia. Since the velocity hardly increases, the relative increase in pressure is reduced.

Consequences

Stable or explosive stellar reactor ?

Why does the Sun not explode all at once, even though all its nuclear fuel is available?
We saw that, for a given mass and composition, it is the opacity κ that determines L
Nuclear reactions adjust to compensate for these surface losses.
This is possible in the case of a perfect gas, where $P = P(\rho, T)$

Case $P = P(\rho, T)$

- Suppose an excess of energy δQ is produced at the center of the star.
- T increases.
- This leads to expansion due to the $P(\rho, T)$ coupling.
- There is an increase in potential energy and consequently a decrease in internal energy (Virial theorem). Therefore, T decreases.
- The nuclear reaction rate decreases, and δQ drops.

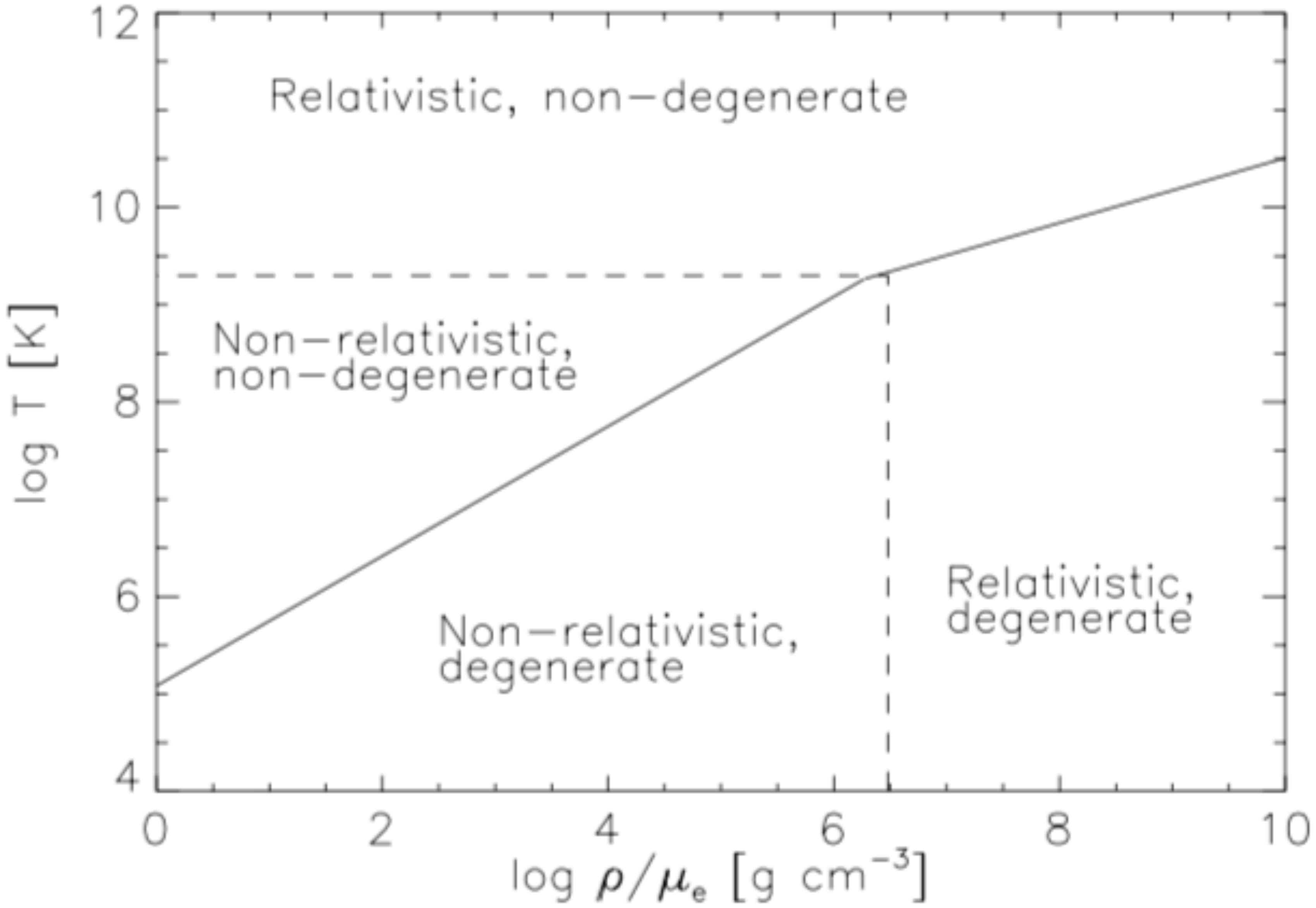
Conversely, if there is an energy deficit, nuclear reactions also adjust.

The system is self-regulated. Nuclear reactions are stable in a star where the perfect gas law prevails.

Case $P = P(\rho)$

- Suppose again an excess of central energy.
- T increases.
- **NO EXPANSION**, because P is independent of T.
- The nuclear production rate increases.
- Therefore, $\delta Q > 0 \Rightarrow \delta Q \gg 0$

No self-regulation: runaway, EXPLOSION. Nuclear reactions are explosive in a degenerate medium.


Relativistic, non-degenerate
Non-relativistic, non-degenerate
Non-relativistic, degenerate
Relativistic, degenerate
log T [K]
log ρ/μe [g cm−3]
12
10
8
6
4
0
2
4
6
8
10

The degeneracy of matter has profound consequences for the physics and evolution of stars.
Let us examine, for example, the relation between the mass and radius of a white dwarf, where the matter is fully degenerate.
The internal pressure can be simply estimated from the equation of hydrostatic equilibrium.

$$\frac{dP}{dr} = -\frac{GM_r\rho(r)}{r^2}$$

To get an order of magnitude, let us take $|\frac{dP}{dr}| = \frac{P_c}{R}$

With the pressure in the central regions $M_r \sim= M/2$, $r \sim R/2$, $\rho \sim 3M/4\pi R^3$

$$\frac{P_c}{R} \simeq -\frac{GM}{R^2}\rho_c \sim \frac{3GM^2}{4\pi R^5}$$

with $P = K_1 \left(\frac{\rho}{\mu_e}\right)^{5/3}$

$$\frac{GM^2}{R^4} \simeq \frac{K_1}{A^{5/3}\mu_e^{5/3}} \frac{M^{5/3}}{R^5}$$

$$R \cdot M^{1/3} \simeq const$$

The greater the mass, the smaller the radius. For a white dwarf with a mass $1M_\odot$, $R \sim 0.01R_\odot$, about the Earth radius

The radius cannot decrease indefinitely; as the density rises, the degeneracy becomes relativistic.

$$P_c \sim 3/2\pi GM^2/R^4 \sim K_2(\frac{\rho}{\mu_e})^{4/3}$$

The mass can only be a constant — this is an upper bound, because if the mass were to increase further, gravity would overcome the Fermi pressure (M^2 vs $M^{4/3}$) and the star would collapse. This is the limiting mass of Chandrasekhar ~1.4 $M_\odot$ for a white dwarf, the limiting mass of Oppenheimer-Volkoff~ 2$M_\odot$ for a neutron star.