

## Solutions to exercice sheet 2

## Fréchet spaces and continuous maps

1. Let  $(X, d)$  be a metric space. Show that

$$\tau_d := \{U \subset X : \forall x \in U, \exists r > 0 \text{ s.t. } B(x, r) \subset U\}$$

defines a topology on  $X$ . Show that if  $(x_k)_{k \in \mathbb{N}}$  is a sequence in  $X$  so that  $\lim_k d(x, x_k) = 0$ , then  $\lim_l d(x_l, x_k) = d(x, x_k)$  for all  $k \in \mathbb{N}$ .

For  $(X, d)$  a Fréchet space, show that for  $n \in \mathbb{N}^*$  and  $r > 0$ ,  $nB(0, r) \subset B(0, nr)$  ( $nB(0, r) := \{ny : y \in B(0, r)\}$ ).

Let  $\{U_1, \dots, U_n\} \subset \tau_d$  and let  $x \in \bigcap_{k=1, \dots, n} U_k$ . Then  $\forall k = 1, \dots, n, \exists r_k > 0$  s.t.  $B(x, r_k) \subset U_k$ . Set  $r := \min\{r_1, \dots, r_n\}$ . Obviously,  $\forall k = 1, \dots, n$  one has  $B(x, r) \subset B(x, r_k) \subset U_k$  and  $B(x, r) \subset \bigcap_{k=1, \dots, n} U_k$ .

Let  $\mathcal{F} \subset \tau_d$  and  $x \in \bigcup_{U \in \mathcal{F}} U$ . By definition, there is then some  $U \in \mathcal{F}$  s.t.  $x \in U$ . Again by definition, there is then some  $r > 0$  s.t.  $B(x, r) \subset U$  and consequently,  $B(x, r) \subset \bigcup_{U \in \mathcal{F}} U$ .

Let  $(x_k)_{k \in \mathbb{N}}$  be a sequence in  $X$  so that  $\lim_k d(x, x_k) = 0$ , and let  $\epsilon > 0$ . Then there is an  $N_\epsilon \in \mathbb{N}$  s.t.  $l \geq N_\epsilon$  implies  $d(x, x_l) < \epsilon$ . Since  $d(x, x_k) \leq d(x, x_l) + d(x_l, x_k)$  and  $d(x_l, x_k) \leq d(x_l, x) + d(x, x_k)$ , one has

$$-\epsilon < -d(x, x_l) \leq d(x_k, x_l) - d(x, x_k) \leq d(x_l, x) < \epsilon$$

and hence, for  $l \geq N_\epsilon$ ,  $|d(x, x_k) - d(x_l, x_k)| < \epsilon$ .

If  $(X, d)$  is Fréchet and if  $y \in B(0, r)$ , then  $d(0, ny) \leq \sum_{k=0}^{n-1} d(ky, (k+1)y) = \sum_{k=0}^{n-1} d(0, y) < nr$ .

2. Let  $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$  be a family of norms on a vector space  $V$ . Show that

$$d(v, w) := \sum_{n \geq 0} 2^{-n} \frac{\|v - w\|_n}{1 + \|v - w\|_n}$$

defines a translation invariant distance on  $V$  (what can you say about the real function  $\mathbb{R}_+ \ni x \mapsto \frac{x}{1+x}$ ?).

Show that in the case of  $V = \mathcal{S}(\mathbb{R}^N)$  one has  $\tau_d = \tau_{\mathcal{S}}$ .

$d$  is obviously positive (note that  $\forall v, w \in V, 0 \leq d(v, w) \leq 2$ ) and  $\forall v, w \in V, d(v, w) = d(w, v)$ . It is also clear that  $d(v, w) = 0 \iff v = w$ . It remains to prove the triangular inequality for  $d$ .

For  $x, y \in \mathbb{R}_+$  we have

$$x \geq y \iff x + xy \geq y + yx \iff \frac{x}{1+x} \geq \frac{y}{1+y}.$$

The map  $\mathbb{R}_+ \ni x \mapsto \frac{x}{1+x}$  is hence increasing. As a consequence, if  $v, w, u \in V$

$$\begin{aligned} d(v, w) &= \sum_{n \geq 0} 2^{-n} \frac{\|v - w\|_n}{1 + \|v - w\|_n} \leq \sum_{n \geq 0} 2^{-n} \frac{\|v - u\|_n + \|u - w\|_n}{1 + \|v - u\|_n + \|u - w\|_n} \\ &= \sum_{n \geq 0} 2^{-n} \frac{\|v - u\|_n}{1 + \|v - u\|_n + \|u - w\|_n} + \sum_{n \geq 0} 2^{-n} \frac{\|u - w\|_n}{1 + \|v - u\|_n + \|u - w\|_n} \\ &\leq \sum_{n \geq 0} 2^{-n} \frac{\|v - u\|_n}{1 + \|v - u\|_n} + \sum_{n \geq 0} 2^{-n} \frac{\|u - w\|_n}{1 + \|u - w\|_n} = d(v, u) + d(u, w). \end{aligned}$$

Let now  $x \in V, r > 0$  and we will show, that there are norms  $\|\cdot\|_k$  for  $k = 1, \dots, N$  and some  $\epsilon > 0$ , so that  $U_{x, \epsilon, 1, \dots, N} \subset B(x, r)$ . Since  $\sum_{k \geq 0} 2^{-k} = 2$ , there is an  $N \in \mathbb{N}$ , so that  $\sum_{k > N} 2^{-k} < \frac{r}{2}$ . Set  $\epsilon := \frac{r}{2(N+1)}$ . For any  $y \in U_{x, \epsilon, 1, \dots, N}$  we thus have

$$\begin{aligned} d(y, x) &= \sum_{0 \leq k \leq N} 2^{-k} \frac{\|x - y\|_k}{1 + \|x - y\|_k} + \sum_{n > N} 2^{-k} \frac{\|x - y\|_n}{1 + \|x - y\|_n} \\ &< \sum_{0 \leq k \leq N} \|x - y\|_k + \sum_{n > N} 2^{-k} < \sum_{0 \leq k \leq N} \|x - y\|_k + \frac{r}{2} < (N+1) \frac{r}{2(N+1)} + \frac{r}{2} = r. \end{aligned}$$

Therefore,  $U_{x, \epsilon, 1, \dots, N} \subset B(x, r)$ . This being true for any  $x \in V$  and any  $r > 0$ , we may conclude, that all open balls  $B(x, r)$  are open for the topology induced by the family of norms  $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$  which is hence finer than  $\tau_d$ .

Reciprocally, consider some  $U_{x, \epsilon, 1, \dots, N}$  for a given  $\epsilon > 0, N \in \mathbb{N}$  and  $x \in V$ . Let  $y \in B(x, r)$ . Then  $r > d(x, y) > 2^{-k} \frac{\|y - x\|_k}{1 + \|y - x\|_k}$ , so that for  $k = 1, \dots, N$  one has  $1 + \|y - x\|_k > 2^{-N} r^{-1} \|y - x\|_k$ , which is to say  $1 > (2^{-N} r^{-1} - 1) \|y - x\|_k$ . If one choses  $r < 2^{-N} \frac{\epsilon}{1 + \epsilon}$ , then  $B(x, r) \subset U_{x, \epsilon, 1, \dots, N}$ . This shows that  $\tau_d$  is finer than the topology induced by the norms  $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ .

3. Let  $q : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$  be the quotient map. Identify any  $z \in S^1$  with some  $x \in \mathbb{R}/\mathbb{Z}$ . This permits one to identify any  $f \in C^\infty(S^1)$  with the periodic smooth function  $f \circ q \in C^\infty(\mathbb{R})$ . For a compact  $K \subset S^1$  let  $C^\infty(S^1 \setminus K) := \{f \in C^\infty(S^1) : \forall \alpha \in \mathbb{N}^N, \partial^\alpha f|_K = 0\}$ . On  $C^\infty(S^1 \setminus K)$  consider the norms

$$p_n(f) := \max_{\alpha \in \mathbb{N}_{\leq n}} \{\|\partial^\alpha f\|_\infty\}.$$

Show that  $(C^\infty(S^1 \setminus K), \{p_n\}_{n \in \mathbb{N}})$  is a Fréchet space.

The fact that  $p_n$  are norms on  $C^\infty(S^1 \setminus K)$  for all  $n \in \mathbb{N}$  is clear. To show completeness of  $C^\infty(S^1)$  under the topology induced by the family of norms  $\{p_n\}_{n \in \mathbb{N}}$ , one may use again the fact that  $C(S^1), S^1$  being compact, is complete for the uniform norm  $\|\cdot\|_\infty$  and then invoke uniform continuity of the Riemann integral.

The fact that  $C^\infty(S^1 \setminus K)$  is complete under the topology induced by the family of norms  $\{p_n\}_{n \in \mathbb{N}}$  follows now from the fact that a sequence of smooth functions  $(f_n)_{n \in \mathbb{N}} \subset C^\infty(S^1 \setminus K)$  can only converge to 0 on  $K$ .

$(C^\infty(S^1 \setminus K), \{p_n\}_{n \in \mathbb{N}})$  is hence complete and the topology may also be induced by the distance  $d(f, g) := \sum_{n \geq 0} 2^{-n} \frac{p_n(f-g)}{1 + p_n(f-g)}$ .

4. Let  $T : \mathcal{S}(\mathbb{R}^N) \rightarrow \mathcal{S}(\mathbb{R}^N)$  be a linear map. Show that the following statements are equivalent:
- (a)  $T$  is continuous
  - (b) For any norm  $\| \cdot \|_n$ , there is a positive constant  $C$  and a norm  $\| \cdot \|_m$ , so that for any  $f \in \mathcal{S}(\mathbb{R}^N)$ ,  $\|T(f)\|_n \leq C\|f\|_m$ .
  - (c) For any sequence  $(f_k)_{k \in \mathbb{N}}$ , if  $\lim_k f_k = 0$  for  $\tau_{\mathcal{S}}$ , then  $\lim_k T(f_k) = 0$  for  $\tau_{\mathcal{S}}$ .
  - (d) For any sequence  $(f_k)_{k \in \mathbb{N}}$ , if  $\lim_k f_k = f \in \mathcal{S}(\mathbb{R}^N)$  for  $\tau_{\mathcal{S}}$ , then  $\lim_k T(f_k) = T(f)$  for  $\tau_{\mathcal{S}}$ .

(a)  $\Rightarrow$  (b) : By continuity of  $T$ , the set  $T^{-1}\{U_{0,1,n}\}$  is open in  $\tau_{\mathcal{S}}$ . Since  $0 \in T^{-1}\{U_{0,1,n}\}$ , there is an  $\epsilon > 0$  and some  $K \in \mathbb{N}$ , so that if  $f \in U_{0,\epsilon,0,1,\dots,K}$ , then  $T(f) \in U_{0,1,n}$ . The norms in  $\mathcal{S}(\mathbb{R}^N)$  being increasing, one has that  $U_{0,\epsilon,K} = U_{0,\epsilon,0,1,\dots,K}$ . Setting therefore  $m = K$  and  $C = \frac{1}{\epsilon}$ , one has that  $\|f\|_m < \frac{1}{C}$  implies  $\|T(f)\|_n < 1$ . Hence,  $\|f\|_m \leq \frac{1}{C}$  implies  $\|T(f)\|_n \leq 1$ , so that for any  $f \in \mathcal{S}(\mathbb{R}^N) \setminus \{0\}$ ,  $\|T(f)\|_n = \|T(C\|f\|_m \frac{f}{C\|f\|_m})\|_n = C\|f\|_m \|T(\frac{f}{C\|f\|_m})\|_n \leq C\|f\|_m$ .

(b)  $\Rightarrow$  (c) : We have to show, that for any norm  $\| \cdot \|_n$ , one has  $\lim_k \|T(f_k)\|_n = 0$  if  $\lim_k f_k = 0$  for  $\tau_{\mathcal{S}}$ . By assumption, there is a positive constant  $C$  and a norm  $\| \cdot \|_m$ , so that  $\|T(f_k)\|_n \leq C\|f_k\|_m$ . But  $\lim_k f_k = 0$  for  $\tau_{\mathcal{S}}$  implies in particular, that  $\lim_k \|f_k\|_m = 0$ . Hence the conclusion.

(c)  $\Rightarrow$  (d) : Since all norms  $\| \cdot \|_m$  are clearly translation invariant,  $\lim_k f_k = f$  is equivalent to  $\lim_k (f - f_k) = 0$  in  $\tau_{\mathcal{S}}$ . By assumption we then have  $\lim_k (T(f - f_k)) = 0$  for  $\tau_{\mathcal{S}}$ , which then is again equivalent to  $\lim_k T(f_k) = T(f)$ .

(d)  $\Rightarrow$  (a) : We prove the contraposition. Suppose  $T$  is not continuous. There is therefore an open set  $U \in \tau_{\mathcal{S}}$  so that  $T^{-1}\{U\} \notin \tau_{\mathcal{S}}$ . Thus, there is an  $f \in T^{-1}\{U\}$ , so that for any open set  $f \ni V \in \tau_{\mathcal{S}}$ ,  $V \not\subset T^{-1}\{U\}$ . In particular, for any  $n \in \mathbb{N}^*$ , there is an  $f_n \in U_{f,n^{-1},n}$  so that  $f_n \notin T^{-1}(U)$ . Clearly,  $\lim_n f_n = f$ , but  $T(f) \in U$  and  $\forall n > 0$ ,  $T(f_n) \notin U$ . This contradicts (d).

5. Show that if the sequences  $(f_k)_{k \in \mathbb{N}}, (g_k)_{k \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^N)$  converge for  $\tau_{\mathcal{S}}$  to  $f$  and  $g$  respectively, then one has  $\lim_k f_k g_k = fg$  and  $\lim_k (f_k + g_k) = f + g$ . Show that if the sequences  $(\varphi_k)_{k \in \mathbb{N}}, (\eta_k)_{k \in \mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^N)$  converge for  $\tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$  to  $\varphi$  and  $\eta$  respectively, then one has  $\lim_k (\varphi_k + \eta_k) = \varphi + \eta$ .

Since the sequences  $(\varphi_k)_{k \in \mathbb{N}}, (\eta_k)_{k \in \mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^N)$  converge for  $\tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$  to  $\varphi$  and  $\eta$  respectively, this implies, that for any fixed  $f \in \mathcal{S}(\mathbb{R}^N)$ , one has

$$\lim_k \varphi_k(f) = \varphi(f) \quad \text{and} \quad \lim_k \eta_k(f) = \eta(f).$$

By linearity of all the functionals in play and of the limit in complex numbers, one has therefore that

$$\lim_k (\varphi_k + \eta_k)(f) = \lim_k \varphi_k(f) + \lim_k \eta_k(f) = \varphi(f) + \eta(f) = (\varphi + \eta)(f).$$

Note first, that for two Schwartz functions  $f$  and  $g$ , one has

$$\begin{aligned} \|fg\|_n &= \max\{\|(1+x \cdot x)^n \partial^\alpha(fg)\|_\infty : \alpha \in \mathbb{N}_{\leq n}^N\} \\ &= \max\{\|(1+x \cdot x)^n \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^\gamma g)\|_\infty : \alpha \in \mathbb{N}_{\leq n}^N\} \\ &\leq \max\{\sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} \|(1+x \cdot x)^n \partial^\beta f\|_\infty \|\partial^\gamma g\|_\infty : \alpha \in \mathbb{N}_{\leq n}^N\} \\ &\leq \binom{N+n}{n} n! \|f\|_n \|g\|_n = \frac{(N+n)!}{N!} \|f\|_n \|g\|_n. \end{aligned}$$

Let  $0 \in U \in \tau_S$  and we shall prove, that for sufficiently large  $k \in \mathbb{N}$ ,  $fg - f_k g_k, f + g - f_k - g_k \in U$ .

Since  $U$  is open and contains 0, there is an  $n \in \mathbb{N}$ , so that  $U_{0,\epsilon,\| \cdot \|_n} \subset U$ .

Since  $\lim_k f_k = f$  and  $\lim_k g_k = g$  for  $\tau_S$ :

- there is a  $k_1 \in \mathbb{N}$  so that  $k \geq k_1$  implies  $\|f - f_k\|_n < \epsilon \frac{N!}{2(\|g\|_n + 1)(N+n)!}$ ,
- there is a  $k_2 \in \mathbb{N}$  so that  $k \geq k_2$  implies  $\|g - g_k\|_n < \epsilon \frac{N!}{2(\|f\|_n + 1)(N+n)!}$ ,
- there is a  $k_3 \in \mathbb{N}$  so that  $k \geq k_3$  implies  $\|g_k\|_n < \|g\|_n + 1$ ,

For  $k \geq \max\{k_1, k_2, k_3\}$  one has

$$\begin{aligned} \|fg - f_k g_k\|_n &= \|fg - f g_k + f g_k - f_k g_k\|_n \leq \|fg - f g_k\|_n + \|f g_k - f_k g_k\|_n \\ &= \|f(g - g_k)\|_n + \|(f - f_k)g_k\|_n \\ &\leq \frac{(N+n)!}{N!} (\|f\|_n \|g - g_k\|_n + \|f - f_k\|_n \|g_k\|_n) < \epsilon, \\ \|f + g - f_k - g_k\|_n &\leq \|f - f_k\|_n + \|g - g_k\|_n < \epsilon. \end{aligned}$$

6. Prove that if  $T : \mathcal{S}(\mathbb{R}^N) \rightarrow \mathcal{S}(\mathbb{R}^N)$  is a continuous linear map, then

$$T^t : \mathcal{S}'(\mathbb{R}^N) \rightarrow \mathcal{S}'(\mathbb{R}^N), \quad (T^t \varphi)(f) := \varphi(Tf)$$

is well-defined and continuous for  $\tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$ .

Prove that for a given  $\alpha \in \mathbb{N}^N$ , the maps

$$\mathcal{S}(\mathbb{R}^N) \ni f \mapsto \partial^\alpha f \in \mathcal{S}(\mathbb{R}^N) \quad \text{and} \quad \mathcal{S}(\mathbb{R}^N) \ni f(x) \mapsto x^\alpha f(x) \in \mathcal{S}(\mathbb{R}^N)$$

are continuous for  $\tau_S$ .

Since  $T : \mathcal{S}(\mathbb{R}^N) \rightarrow \mathcal{S}(\mathbb{R}^N)$  and if  $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ , then  $T^t(\varphi) = \varphi \circ T$  is certainly a linear map from  $\mathcal{S}(\mathbb{R}^N)$  to  $\mathcal{S}(\mathbb{R}^N)$ .

If  $U$  is an open set in  $\mathbb{C}$ , then  $(T^t(\varphi))^{-1}\{U\} = (\varphi \circ T)^{-1}\{U\} = T^{-1}\{\varphi^{-1}\{U\}\}$ . Since  $\varphi$  is a tempered distribution,  $\varphi^{-1}\{U\} \in \tau_S$  and since  $T$  is continuous,  $T^{-1}\{\varphi^{-1}\{U\}\} \in \tau_S$  again. Hence,  $T^t(\varphi)$  is a continuous linear map on  $\mathcal{S}(\mathbb{R}^N)$  for  $\tau_S$ .

$T$  is then indeed a linear map from  $\mathcal{S}'(\mathbb{R}^N)$  to  $\mathcal{S}'(\mathbb{R}^N)$ . Let us show that it is also continuous for  $\tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$ .

Let  $U \in \tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$  and let  $\varphi \in (T^t)^{-1}\{U\}$ . Since  $U$  is open, there are Schwartz functions  $f_1, \dots, f_n$ , so that  $U_{T^t(\varphi), f_1, \dots, f_n} = \{\eta \in \mathcal{S}'(\mathbb{R}^N) : \forall k = 1, \dots, n, |\eta(f_k) - T^t(\varphi)(f_k)| < 1\} \subset U$ .

Define then  $h_k := T(f_k)$  for all  $k = 1, \dots, n$  and consider the open set  $U_{\varphi, h_1, \dots, h_n} \in \tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$ . If  $\psi \in U_{\varphi, h_1, \dots, h_n}$ , then  $\forall k = 1, \dots, n, |\psi(h_k) - \varphi(h_k)| < 1$ , so that  $T^t(\psi) \in U_{T^t(\varphi), f_1, \dots, f_n}$ . Hence,  $U_{\varphi, h_1, \dots, h_n} \subset (T^t)^{-1}\{U_{T^t(\varphi), f_1, \dots, f_n}\} \subset (T^t)^{-1}\{U\}$ , which shows, that the latter is open.

For  $n \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^N$ , we have that

$$\begin{aligned} \|\partial^\alpha f\|_n &= \max\{\|(1+x \cdot x)^n \partial^{\alpha+\beta} f\|_\infty : \beta \in \mathbb{N}_{\leq n}^N\} \\ &\leq \max\{\|(1+x \cdot x)^{n+|\alpha|} \partial^\beta f\|_\infty : \beta \in \mathbb{N}_{\leq n+|\alpha|}^N\} = \|f\|_{n+|\alpha|}, \\ \|x^\alpha f\|_n &= \max\{\|(1+x \cdot x)^n \partial^\beta (x^\alpha f)\|_\infty : \beta \in \mathbb{N}_{\leq n}^N\} \\ &\leq \max\left\{\sum_{\gamma+\delta=\beta} \binom{\beta}{\gamma} \|(1+x \cdot x)^n (\partial^\gamma x^\alpha)(\partial^\delta f)\|_\infty : \beta \in \mathbb{N}_{\leq n}^N\right\} \\ &= \max\left\{\sum_{\gamma+\delta=\beta} \binom{\beta}{\gamma} \frac{\alpha!}{(\alpha-\gamma)!} \|(1+x \cdot x)^n (x^{\alpha-\gamma})(\partial^\delta f)\|_\infty : \beta \in \mathbb{N}_{\leq n}^N\right\} \\ &\leq \max\left\{\sum_{\gamma+\delta=\beta} \binom{\beta}{\gamma} (\alpha!) \|(1+x \cdot x)^{n+|\alpha|} (\partial^\delta f)\|_\infty : \beta \in \mathbb{N}_{\leq n}^N\right\} \\ &\leq \max\left\{\sum_{\gamma \leq \beta} \binom{\beta}{\gamma} (\alpha!) \|f\|_{n+|\alpha|} : \beta \in \mathbb{N}_{\leq n}^N\right\} \leq \binom{N+n}{n} 2^n (\alpha!) \|f\|_{n+|\alpha|}. \end{aligned}$$

7. The Poincaré group is defined as the set of couples  $(\Lambda, d) \in \mathbb{M}_4(\mathbb{R}^4) \times \mathbb{R}^4$ , so that  $\Lambda^t \eta \Lambda = \eta$ , where  $\eta = \text{diag}(1, -1, -1, -1)$ . The group product is defined as  $(\Lambda_1, d_1) \cdot (\Lambda_2, d_2) := (\Lambda_1 \Lambda_2, d_1 + \Lambda_1 d_2)$ .

Define the action of this group on  $\mathcal{S}(\mathbb{R}^4)$  by

$$\mathcal{S}(\mathbb{R}^4) \ni f(x) \mapsto ((\Lambda, d) \circ f)(x) := f((\Lambda, d)^{-1}x)$$

and show that this action is continuous for  $\tau_{\mathcal{S}}$ .

(Hint: for a linear map  $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}^N$  and a function  $f \in C^\infty(\mathbb{R}^N)$ , Faà di Bruno's formula reads

$$\partial^\alpha (f \circ \varphi) = \sum_{\beta \in \mathbb{N}_{=|\alpha|}^N} (\partial^\beta f)(\varphi(x)) \sum_{\substack{\gamma_1, \dots, \gamma_N \in \mathbb{N}^N, \\ \forall 1 \leq j \leq N, |\gamma_j| = \alpha_j, \\ \sum_{j=1}^N \gamma_j = \beta}} \alpha! \prod_{j=1}^N \frac{1}{\gamma_j!} \left(\frac{\partial \varphi(x)}{\partial x_j}\right)^{\gamma_j}.$$

)

For a given  $x \in \mathbb{R}^4$  and a fixed element  $(\Lambda, d)$  in the Poincaré group, let us denote

$x \cdot x$  by  $|x|^2$  and  $\sup\{|\Lambda x| : |x| \leq 1\}$  by  $\|\Lambda\|$ . One then has

$$\begin{aligned} & 1 + (\Lambda x + d) \cdot (\Lambda x + d) = 1 + |\Lambda x + d|^2 \\ & \leq 1 + (\|\Lambda\| |x| + |d|)^2 = 1 + \|\Lambda\|^2 x \cdot x + 2\|\Lambda\| |x| |d| + |d|^2 \\ & \leq 1 + x \cdot x + \|\Lambda\|^2(1 + x \cdot x) + \|\Lambda\| |d|(1 + x \cdot x) + |d|^2(1 + x \cdot x) \\ & \leq (1 + |d| + \|\Lambda\|)^2(1 + x \cdot x). \end{aligned}$$

Also, for a fixed  $\alpha \in \mathbb{N}^4$ , one has

$$\begin{aligned} \partial^\alpha ((\Lambda, d) \circ f) &= \partial^\alpha (f((\Lambda, d)^{-1}x)) = \partial^\alpha (f(\Lambda^{-1}x - \Lambda^{-1}d)) \\ &= \sum_{\beta \in \mathbb{N}_{=|\alpha}^4} (\partial^\beta f)(\Lambda^{-1}x - \Lambda^{-1}d) \sum_{\substack{\gamma_1, \dots, \gamma_4 \in \mathbb{N}^4, \\ \forall 1 \leq j \leq 4, |\gamma_j| = \alpha_j, \\ \sum_{j=1}^4 \gamma_j = \beta}} \alpha! \prod_{j=1}^4 \frac{1}{\gamma_j!} \left(\frac{\partial \Lambda x}{\partial x_j}\right)^{\gamma_j}. \end{aligned}$$

Therefore, if  $\alpha \in \mathbb{N}_{\leq n}^4$ ,

$$\begin{aligned} & |(1 + x \cdot x)^n \partial^\alpha ((\Lambda, d) \circ f)| \\ &= |(1 + x \cdot x)^n \sum_{\beta \in \mathbb{N}_{=|\alpha}^4} (\partial^\beta f)(\Lambda^{-1}x - \Lambda^{-1}d) \sum_{\substack{\gamma_1, \dots, \gamma_4 \in \mathbb{N}^4, \\ \forall 1 \leq j \leq 4, |\gamma_j| = \alpha_j, \\ \sum_{j=1}^4 \gamma_j = \beta}} \alpha! \prod_{j=1}^4 \frac{1}{\gamma_j!} \left(\frac{\partial \Lambda x}{\partial x_j}\right)^{\gamma_j}| \\ &\leq \alpha! (1 + x \cdot x)^n \left( \sum_{\beta \in \mathbb{N}_{=|\alpha}^4} |(\partial^\beta f)(\Lambda^{-1}x - \Lambda^{-1}d)| \sum_{\substack{\gamma_1, \dots, \gamma_4 \in \mathbb{N}^4, \\ \forall 1 \leq j \leq 4, |\gamma_j| = \alpha_j, \\ \sum_{j=1}^4 \gamma_j = \beta}} \|\Lambda\|^{|\beta|} \right) \\ &\leq \alpha! (1 + x \cdot x)^n \left( \sum_{\beta \in \mathbb{N}_{=|\alpha}^4} |(\partial^\beta f)(\Lambda^{-1}x - \Lambda^{-1}d)| \prod_{j=1}^4 \binom{4 + \alpha_j - 1}{\alpha_j} \|\Lambda\|^{|\beta|} \right) \\ &\leq \alpha! \binom{\bar{3} + \alpha}{\alpha} \|\Lambda\|^{|\alpha|} (1 + x \cdot x)^n \sum_{\beta \in \mathbb{N}_{=|\alpha}^4} |(\partial^\beta f)(\Lambda^{-1}x - \Lambda^{-1}d)| \\ &\stackrel{y = \Lambda^{-1}x - \Lambda^{-1}d}{=} \alpha! \binom{\bar{3} + \alpha}{\alpha} \|\Lambda\|^{|\alpha|} (1 + |\Lambda y + d|^2)^n \sum_{\beta \in \mathbb{N}_{=|\alpha}^4} |(\partial^\beta f)(y)| \\ &\leq \alpha! \binom{\bar{3} + \alpha}{\alpha} \|\Lambda\|^{|\alpha|} (1 + \|\Lambda\| + |d|)^{2n} \sum_{\beta \in \mathbb{N}_{=|\alpha}^4} |(1 + y \cdot y)^n (\partial^\beta f)(y)| \\ &\leq ((3 + n)!)^5 \|\Lambda\|^n (1 + \|\Lambda\| + |d|)^{2n} \|f\|_n, \end{aligned}$$

and taking the supremum over all  $x \in \mathbb{R}^4$  shows continuity.

8. Let  $f, g \in \mathcal{S}(\mathbb{R}^N)$  and let  $h \in \mathcal{L}^1(\mathbb{R}^N, \mu_L)$ . Show that  $f * h \in C^\infty(\mathbb{R}^N)$  and that  $f * g \in \mathcal{S}(\mathbb{R}^N)$  again.

(Hint: for differentiability, you might wanna use dominated convergence and Rolle's theorem.)

Prove then that for  $\alpha \in \mathbb{N}^N$ ,  $\partial^\alpha(f * g) = (\partial^\alpha f) * g$  and that  $\mathcal{S}(\mathbb{R}^N) \ni g \mapsto f * g$  is continuous for  $\tau_{\mathcal{S}}$ .

$\int_{\mathbb{R}^N} f(x-y)h(y)\mu_L(dy)$  is certainly well-defined for any fixed  $x \in \mathbb{R}^N$  since  $f(x-y) \in \mathcal{S}(\mathbb{R}^N)$ , implying  $f(x-y)h(y) \in \mathcal{L}^1(\mathbb{R}^N, \mu_L)$ .

Moreover, by Rolle's theorem, for any  $x, x' \in \mathbb{R}^N$ , there is a number  $0 < c < 1$ , so that  $f(x-y)h(y) - f(x'-y)h(y) = (J_f(x'-y + c(x-x')) \cdot (x-x'))h(y)$ , where  $J_f(x)$  is the gradient of  $f$ .

For  $(x'-x) \cdot (x'-x) \leq 1$ , this then shows, that both  $f(x-y)h(y) - f(x'-y)h(y)$  and  $|x'-x|^{-1}(f(x-y) - f(x'-y))h(y)$  are absolutely bounded by  $N\|f\|_1|h(y)|$ .

Applying dominated convergence, one gets

$$\lim_{x' \rightarrow x} (f * h)(x') = (f * h)(x) \quad \text{and}$$

$$\forall \delta \in \mathbb{N}_{=1}^N, \quad \lim_{a \rightarrow 0} \frac{1}{a} ((f * h)(x + a\delta) - (f * h)(x)) = \left( (\partial^{\bar{1}\delta} f) * h \right) (x).$$

Since  $\partial^{\bar{1}\delta} f$  is again a Schwartz function, this shows continuity and differentiability of the function  $f * h$ . By induction, this shows also that  $f * h$  is smooth and that for any multi-index  $\alpha$ ,  $\partial^\alpha(f * h) = (\partial^\alpha f) * h$ .

If in addition  $g \in \mathcal{S}(\mathbb{R}^N)$ , then for any fixed  $x \in \mathbb{R}^N$ , one has

$$\begin{aligned} |(1+x \cdot x)^n (f * g)(x)| &= \left| \int_{\mathbb{R}^N} (1+x \cdot x)^n f(x-y)g(y)\mu_L(dy) \right| \\ &\leq \int_{\mathbb{R}^N} (1+x \cdot x)^n |f(x-y)g(y)|\mu_L(dy) \\ &\leq \int_{\mathbb{R}^N} (1+|x|)^{2n} |f(x-y)g(y)|\mu_L(dy) \\ &\leq \int_{\mathbb{R}^N} (1+|x-y|)^{2n} (1+|y|)^{2n} |f(x-y)g(y)|\mu_L(dy) \\ &\leq \int_{\mathbb{R}^N} 2^n (1+(x-y) \cdot (x-y))^{2n} 2^n (1+y \cdot y)^n |f(x-y)g(y)|\mu_L(dy) \\ &\leq 4^n \|(1+x \cdot x)^n f(x)\|_{L^2} \|(1+x \cdot x)^n g(x)\|_{L^2} \leq 4^n \|f\|_{2n} \|g\|_{2n}, \end{aligned}$$

which shows that  $(1+x \cdot x)^n (f * g)(x)$  has to remain bounded for any  $n \in \mathbb{N}$ . Hence,  $f * g \in \mathcal{S}(\mathbb{R}^N)$ .

But it also shows continuity, since for all  $n \in \mathbb{N}$  and any  $\alpha \in \mathbb{N}_{\leq n}^N$ ,

$$\|f * g - f * h\|_n = \|f * (g - h)\|_n \leq \binom{N+n}{n} 4^n \|f\|_{2n} \|g - h\|_{2n}.$$

9. For a given Schwartz function  $f$ , show that for any  $n \in \mathbb{N}$ , there is a constant  $C_{n,f}$  and a function  $h(y)$  with  $\lim_{|y| \rightarrow 0} h(y) = 0$ , so that

$$\|(1 + x \cdot x)^n (f(x + y) - f(x))\|_\infty \leq C_{n,f} h(y).$$

Let  $(d_n(x))_{n \in \mathbb{N}}$  be a Dirac sequence. Use the previous estimation to show that

$$\forall n \in \mathbb{N}, \quad \lim_k \|f - d_k * f\|_n = 0.$$

For  $f \in \mathcal{S}(\mathbb{R}^N)$  and  $x, y \in \mathbb{R}^N$ , one has by Rolle's theorem

$$|(1 + x \cdot x)^n (f(x + y) - f(x))| = |(1 + x \cdot x)^n (\nabla f)(x + ty) \cdot y|$$

for some  $t \in ]0, 1[$  and where  $(\nabla f)$  is the gradient of  $f$ . Now, if  $|x| = x \cdot x^{1/2}$ , then  $|x + y| \leq |x| + |y|$  and  $2|x| \leq 1 + x \cdot x$ , so that

$$\begin{aligned} (1 + x \cdot x) &= (1 + |x + ty - ty|^2) \leq (1 + |x + ty|^2 + 2|x + ty| |ty| + |ty|^2) \\ &\leq (1 + |x + ty|^2 + (1 + |x + ty|^2) |ty| + |ty|^2) \leq 2(1 + |x + ty|^2)(1 + |ty|)^2. \end{aligned}$$

Therefore, for some  $t \in ]0, 1[$ ,

$$\begin{aligned} |(1 + x \cdot x)^n (f(x + y) - f(x))| &\leq 2^n |(1 + |x + ty|^2)^n (\nabla f)(x + ty) \cdot y (1 + |y|^2)^n| \\ &\leq 2^n N \|f\|_n |y| (1 + |y|^2)^n. \end{aligned}$$

Hence,  $\|f(x + y) - f(x)\|_n \leq 2^n \|f\|_{n+1} |y| (1 + |y|^2)$ , which shows, that  $\lim_{|y| \rightarrow 0} (f(x + y) - f(x)) = 0$  for  $\tau_S$ .

For  $f \in \mathcal{S}(\mathbb{R}^N)$  one has

$$\begin{aligned} |f(x) - (d_n * f)(x)| &= |f(x) - \int_{\mathbb{R}^N} d_n(x - y) f(y) \mu_L(dy)| \\ &= |f(x) \int_{\mathbb{R}^N} d_n(y) \mu_L(dy) - \int_{\mathbb{R}^N} d_n(y) f(x - y) \mu_L(dy)| \\ &= \left| \int_{\mathbb{R}^N} d_n(y) (f(x) - f(x - y)) \mu_L(dy) \right|. \end{aligned}$$

Since  $(d_k)_{k \in \mathbb{N}}$  is a Dirac sequence, there is an  $n_r \in \mathbb{N}$ , so that  $k \geq n_r$  implies  $\text{supp}(d_k) \subset D(0, r)$ . For such a  $k \in \mathbb{N}$ , one therefore has

$$\begin{aligned} |(1 + x \cdot x)^n (f(x) - (d_k * f)(x))| &\leq \int_{\mathbb{R}^N} (1 + x \cdot x)^n d_k(y) |f(x) - f(x - y)| \mu_L(dy) \\ &= \int_{D(0,r)} d_k(y) (1 + x \cdot x)^n |f(x) - f(x - y)| \mu_L(dy) \\ &\leq r^{4n} \|f\|_{n+1} \int_{D(0,r)} d_k(y) \mu_L(dy) = r^{4n} \|f\|_{n+1}. \end{aligned}$$

Consequently,  $k \geq n_r$  implies  $\|(1 + x \cdot x)^n (f - d_k * f)\|_\infty \leq r^{4n} \|f\|_{n+1}$ , which converges to 0.

For a given  $n \in \mathbb{N}$  and  $\alpha \in \mathbb{N}_{\leq n}^N$ , one has that  $\partial^\alpha f$  is again a Schwartz function, so that here also, there is an  $n_\alpha$ , so that  $k \geq n_\alpha$  implies

$$\begin{aligned} \|(1+x \cdot x)^n (\partial^\alpha f - \partial^\alpha (d_k * f))\|_\infty &= \|(1+x \cdot x)^n (\partial^\alpha f - d_k * \partial^\alpha f)\|_\infty \\ &\leq r4^n \|\partial^\alpha f\|_{n+1} \leq r4^n \|f\|_{2n+1}. \end{aligned}$$

Since there are finitely many such  $\alpha$ 's in  $\mathbb{N}_{\leq n}^N$ , one may take  $N_r = \max\{n_\alpha : \alpha \in \mathbb{N}_{\leq n}^N\}$  and for  $k \geq N_r$ , one has

$$\|f - d_k * f\|_n \leq r4^n \|f\|_{2n+1}.$$

Therefore,  $\lim_k d_k * f = f$  for  $\tau_S$ .

10. Let  $(d_n(x))_{n \in \mathbb{N}}$  be a Dirac sequence. Prove that the functionals

$$\mathbb{S}(\mathbb{R}^N) \ni \delta_n(f) := \int_{\mathbb{R}^N} (d_n * f) \mu_L(dx)$$

converge in the weak\* topology to  $\varphi_1$ .

Show that for  $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ ,  $(d_n * \varphi)_{n \in \mathbb{N}}$  defines a sequence of  $C^\infty$ -functions converging to  $\varphi$  for the topology  $\tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$ .

Since we clearly have  $d_n \in \mathcal{S}(\mathbb{R}^N)$  for all  $n \in \mathbb{N}$ , the map  $f \mapsto d_n^* f$  is continuous for  $\tau_S$  and for each fixed  $n \in \mathbb{N}$ . Hence,  $(\delta_n)_{n \in \mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^N)$ .

Furthermore, for  $f \in \mathcal{S}(\mathbb{R}^N)$ , then  $d_n(y)f(x-y)$  is clearly in  $\mathcal{S}(\mathbb{R}^{2N})$ , so that one may apply Fubini's theorem to obtain

$$\begin{aligned} \delta_n(f) &= \int_{\mathbb{R}^N} (d_n * f)(x) \mu_L(dx) = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} d_n(x-y) f(y) \mu_L(dy) \right) \mu_L(dx) \\ &= \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} d_n(y) f(x-y) \mu_L(dy) \right) \mu_L(dx) \\ &= \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} d_n(y) f(x-y) \mu_L(dx) \right) \mu_L(dy) \\ &= \int_{\mathbb{R}^N} d_n(y) \varphi_1(f) \mu_L(dx) = \varphi_1(f). \end{aligned}$$

It is then clear that  $(\delta_n)_{n \in \mathbb{N}}$  converges \*-weakly to  $\varphi_1$ .

By the previous exercise, if  $f \in \mathcal{S}(\mathbb{R}^N)$  then  $\lim_k d_k * f = f$  for the topology  $\tau_S$ . Since  $(P^t d_k)_{k \in \mathbb{N}}$  is again a Dirac sequence, one has  $\lim_k P^t(d_k) * f = f$  as well. If  $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ , then by continuity one has  $\lim_k (d_k * \varphi)(f) = \lim_k \varphi(P^t(d_k) * f) = \varphi(\lim_k P^t(d_k) * f) = \varphi(f)$ . Therefore,  $\lim_k (d_k * \varphi) = \varphi$  for  $\tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$ .

It remains to show, that for each  $k \in \mathbb{N}$ ,  $(d_k * \varphi)(f)$  is integration of  $f$  against a  $C^\infty$ -function  $g_k$ . By Schwartz representation theorem, there is a polynomially bounded function  $g \in C(\mathbb{R}^N)$  and a multi-index  $\alpha$ , so that for any Schwartz function  $f$ ,

$$\varphi(f) = \int_{\mathbb{R}^N} g \partial^\alpha f \mu_L(dx).$$

Therefore,

$$\begin{aligned}
(d_k * \varphi)(f) &= \varphi((P^t d_k) * f) = \int_{\mathbb{R}^N} g \partial^\alpha ((P^t d_k) * f) \mu_L(dx) \\
&= \int_{\mathbb{R}^N} g(x) \partial^\alpha \left( \int_{\mathbb{R}^N} d_k(y-x) f(y) \mu_L(dy) \right) \mu_L(dx) \\
&= \int_{\mathbb{R}^N} g(x) \left( \int_{\mathbb{R}^N} d_k(y-x) \partial^\alpha f(y) \mu_L(dy) \right) \mu_L(dx).
\end{aligned}$$

The function  $g(x)$  being polynomially bounded and continuous, there is an  $m \in \mathbb{N}$ , so that  $g(x)(1+x \cdot x)^{-m} \in \mathcal{L}^1(\mathbb{R}^N, \mu_L)$ . Furthermore,

$$\begin{aligned}
|g(x) d_k(y-x) \partial^\alpha f(y)| &= |g(x)(1+x \cdot x)^{-m} (1+x \cdot x)^m d_k(x-y) \partial^\alpha f(y)| \\
&\leq |g(x)(1+x \cdot x)^{-m} (1+|x|)^{2m} d_k(x-y) \partial^\alpha f(y)| \\
&\leq |g(x)(1+x \cdot x)^{-m} (1+|x-y|+|y|)^{2m} d_k(x-y) \partial^\alpha f(y)| \\
&\leq |g(x)(1+x \cdot x)^{-m} (1+|x-y|)^{2m} d_k(x-y) (1+|y|)^{2m} \partial^\alpha f(y)| \\
&\leq 4^{2m} |g(x)(1+x \cdot x)^{-m} d_k(x-y) (1+y \cdot y)^m \partial^\alpha f(y)|,
\end{aligned}$$

which is in  $\mathcal{L}^1(\mathbb{R}^{2N}, \mu_L)$ , since  $(1+y \cdot y)^m \partial^\alpha f(y)$  is again a Schwartz function. Therefore, we may apply Fubini's theorem and obtain

$$\begin{aligned}
(d_k * \varphi)(f) &= \int_{\mathbb{R}^N} g(x) \left( \int_{\mathbb{R}^N} d_k(y-x) \partial^\alpha f(y) \mu_L(dy) \right) \mu_L(dx) \\
&= \int_{\mathbb{R}^N} \partial^\alpha f(y) \left( \int_{\mathbb{R}^N} d_k(y-x) g(x) \mu_L(dx) \right) \mu_L(dy).
\end{aligned}$$

The integral  $\int_{\mathbb{R}^N} d_k(y-x) g(x) \mu_L(dx)$  results in a function  $h_k(y)$  and by using again a Rolle-type argument can be shown to be  $C^\infty$ . Partial integration permits one to conclude, that the  $C^\infty$ -function  $g_k(y) = (-1)^{|\alpha|} \partial^\alpha h_k(y)$  satisfies

$$(d_k * \varphi)(f) = \int_{\mathbb{R}^N} g_k f d\mu_L.$$

**Remark:** The sequence of  $C^\infty$ -functions  $(g_k)_{k \in \mathbb{N}}$  is sometimes called a **regularisation** of  $\varphi$ . This sequence however does only converge in the distributional sense (i.e. in the weak\* topology) to  $\varphi$ .