Exercice sheet 2

Fréchet spaces and continuous maps

1. Let (X, d) be a metric space. Show that

$$\tau_d := \{U \subset X \,:\, \forall x \in U,\, \exists r > 0 \text{ s.t. } B(x,r) \subset U\}$$

defines a topology on X. Show that if $(x_k)_{k\in\mathbb{N}}$ is a sequence in X so that $\lim_k d(x, x_k) = 0$, then $\lim_l d(x_l, x_k) = d(x, x_k)$ for all $k \in \mathbb{N}$.
For (X, d) a Fréchet space, show that for $n \in \mathbb{N}^*$ and $r > 0$, $nB(0, r) \subset B(0, nr)$ ($nB(0, r) := \{ny \,:\, y \in B(0, r)\}$).

2. Let $\{\|\|_n\}_{n\in\mathbb{N}}$ be a family of norms on a vector space V. Show that

$$d(v, w) := \sum_{n\geq 0} 2^{-n} \frac{\|v - w\|_n}{1 + \|v - w\|_n}$$

defines a translation invariant distance on V (what can you say about the real function $\mathbb{R}_+ \ni x \mapsto \frac{x}{1+x}$?).
Show that in the case of $V = \mathcal{S}(\mathbb{R}^N)$ one has $\tau_d = \tau_{\mathcal{S}}$.

3. Let $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map. Identify any $z \in S^1$ with some $x \in \mathbb{R}/\mathbb{Z}$. This permits one to identify any $f \in C^\infty(S^1)$ with the periodic smooth function $f \circ q \in C^\infty(\mathbb{R})$.
For a compact $K \subset S^1$ let $C^\infty(S^1 \setminus K) := \{f \in C^\infty(S^1) \,:\, \forall\alpha \in \mathbb{N}^N,\, \partial^\alpha f\big|_K = 0\}$.
On $C^\infty(S^1 \setminus K)$ consider the norms

$$p_n(f) := \max\nolimits_{\alpha\in\mathbb{N}^N_{\leq n}} \{\|\partial^\alpha f\|_\infty\}.$$

Show that $(C^\infty(S^1 \setminus K), \{p_n\}_{n\in\mathbb{N}})$ is a Fréchet space.

4. Let $T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ be a linear map. Show that the following statements are equivalent:

(a) T is continuous

(b) For any norm $\|\ \|_n$, there is a positive constant C and a norm $\|\|_m$, so that for any $f \in \mathcal{S}(\mathbb{R}^N)$, $\|T(f)\|_n \leq C\|f\|_m$.

(c) For any sequence $(f_k)_{k\in\mathbb{N}}$, if $\lim_k f_k = 0$ for $\tau_{\mathcal{S}}$, then $\lim_k T(f_k) = 0$ for $\tau_{\mathcal{S}}$.

(d) For any sequence $(f_k)_{k\in\mathbb{N}}$, if $\lim_k f_k = f \in \mathcal{S}(\mathbb{R}^N)$ for $\tau_{\mathcal{S}}$, then $\lim_k T(f_k) = T(f)$ for $\tau_{\mathcal{S}}$.

5. Show that if the sequences $(f_k)_{k\in\mathbb{N}}, (g_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^N)$ converge for $\tau_{\mathcal{S}}$ to f and g respectively, then one has $\lim_k f_k g_k = fg$ and $\lim_k (f_k + g_k) = f + g$.
Show that if the sequences $(\varphi_k)_{k\in\mathbb{N}}, (\eta_k)_{k\in\mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^N)$ converge for $\tau(\mathcal{S}'(\mathbb{R}^N, \mathcal{S}(\mathbb{R}^N))$ to φ and η respectively, then one has $\lim_k(\varphi_k + \eta_k) = \varphi + \eta$.

6. Prove that if $T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is a continuous linear map, then

$$T^t : \mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N), \quad (T^t\varphi)(f) := \varphi(Tf)$$

is well-defined and continuous for $\tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$.
Prove that for a given $\alpha \in \mathbb{N}^N$, the maps

$$\mathcal{S}(\mathbb{R}^N) \ni f \mapsto \partial^\alpha f \in \mathcal{S}(\mathbb{R}^N) \quad \text{and} \quad \mathcal{S}(\mathbb{R}^N) \ni f(x) \mapsto x^\alpha f(x) \in \mathcal{S}(\mathbb{R}^N)$$

are continuous for $\tau_{\mathcal{S}}$.

7. The Poincaré group is defined as the set of couples $(\Lambda, d) \in \mathbb{M}_4(\mathbb{R}^4) \times \mathbb{R}^4$, so that $\Lambda^t \eta \Lambda = \eta$, where $\eta = \mathrm{diag}(1, -1, -1, -1)$. The group product is defined as $(\Lambda_1, d_1) \cdot (\Lambda_2, d_2) := (\Lambda_1\Lambda_2, d_1 + \Lambda_1 d_2)$.
Define the action of this group on $\mathcal{S}(\mathbb{R}^4)$ by

$$\mathcal{S}(\mathbb{R}^4) \ni f(x) \mapsto ((\Lambda, d) \circ f)(x) := f((\Lambda, d)^{-1}x)$$

and show that this action is continuous for $\tau_{\mathcal{S}}$.
(Hint: for a linear map $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ and a function $f \in C^\infty(\mathbb{R}^N)$, Faà di Bruno's formula reads

$$\partial^\alpha (f \circ \varphi) = \sum_{\beta \in \mathbb{N}^N_{=|\alpha|}} (\partial^\beta f)(\varphi(x)) \sum_{\substack{\gamma_1, \ldots, \gamma_N \in \mathbb{N}^N, \\ \forall 1 \le j \le N,\, |\gamma_j| = \alpha_j, \\ \sum_{j=1}^N \gamma_j = \beta}} \alpha! \prod_{j=1}^N \frac{1}{\gamma_j!} \left(\frac{\partial \varphi(x)}{\partial x_j}\right)^{\gamma_j}.$$

)

8. Let $f, g \in \mathcal{S}(\mathbb{R}^N)$ and let $h \in \mathcal{L}^1(\mathbb{R}^N, \mu_L)$.Show that $f * h \in C^\infty(\mathbb{R}^N)$ and that $f * g \in \mathcal{S}(\mathbb{R}^N)$ again.
(Hint: for differentiability, you might wanna use dominated convergence and Rolle's theorem.)
Prove then that for $\alpha \in \mathbb{N}^N$, $\partial^\alpha (f * g) = (\partial^\alpha f) * g$ and that $\mathcal{S}(\mathbb{R}^N) \ni g \mapsto f * g$ is continuous for $\tau_{\mathcal{S}}$.

9. For a given Schwartz function f, show that for any $n \in \mathbb{N}$, there is a constant $C_{n,f}$ and a function $h(y)$ with $\lim_{|y| \to 0} h(y) = 0$, so that

$$\|(1 + x \cdot x)^n (f(x + y) - f(x))\|_\infty \le C_{n,f} h(y).$$

Let $(d_n(x))_{n \in \mathbb{N}}$ be a Dirac sequence. Use the previous estimation to show that

$$\forall n \in \mathbb{N}, \quad \lim_k |||f - d_k * f|||_n = 0.$$

10. Let $(d_n(x))_{n \in \mathbb{N}}$ be a Dirac sequence. Prove that the functionals

$$\mathbb{S}(\mathbb{R}^N) \ni \mapsto \delta_n(f) := \int_{\mathbb{R}^N} (d_n * f) \mu_L(dx)$$

converge in the weak* topology to φ_1.
Show that for $\varphi \in \mathcal{S}'(\mathbb{R}^N)$, $(d_n * \varphi)_{n \in \mathbb{N}}$ defines a sequence of C^∞-functions converging to φ for the topology $\tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$.