Exercice sheet 1

Schwartz space and tempered distributions

1. Prove the Riesz & Fréchet theorem, namely, that given a Hilbert space H, the anti-linear map
$$J:\ H \ni x \mapsto \langle x,\ \ \rangle =: J(x) \in H'$$
is an isometric isomorphism.
(Hint: if $V \subset H$ is a closed subspace, then $H = V \oplus V^{\perp}$.)

2. Let H be a Hilbert space and consider a Banach space B. Prove that any continuous and linear map $\xi : D \to B$, defined on a dense set $D \subset H$ has a unique and isometric extension $\overline{\xi} : H \to B$.

3. Let V be a $\mathbb{K}-$vector space and let $\|\ \ \|_{1,2} : V \to \mathbb{R}_+$ be two norms:

(a) Show that if $\forall x \in V$, $\|x\|_1 \leq \|x\|_2$, then $\{x \in V\ :\ \|x\|_2 < 1\} \subset \{x \in V\ :\ \|x\|_1 < 1\}$.

(b) Show that the topology τ_2 defined by $\|\ \ \|_2$ is finer than the topology τ_1 defined by $\|\ \ \|_1$, i.e. $\tau_1 \subset \tau_2$. Prove as a consequence, that any sequence $(x_n)_{n\in\mathbb{N}} \subset V$ converges for τ_1 if it does so for τ_2.

(c) Show that if W is a $\mathbb{K}-$vector space with a topology τ_W and if $f : V \to W$ is a continuous map with respect to τ_1, then f is also continuous with respect to τ_2.

(d) Show that if in addition, there is a positive constant C so that $\forall x \in V$, $\|x\|_2 \leq C\|x\|_1$, then $\tau_1 = \tau_2$.

4. (a) For $f, g \in C^n(\mathbb{R}^N)$ and $\alpha \in \mathbb{N}^N$ with $|\alpha| \leq n$, show that
$$\partial^\alpha (fg)(x) = \sum_{\substack{\beta,\gamma\in\mathbb{N}^N,\\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \partial^\beta f(x) \partial^\gamma g(x).$$

(b) Show that the cardinality of the set $\mathbb{N}^N_{\leq n} := \{\alpha \in \mathbb{N}^N\ :\ |\alpha| \leq n\}$ is $\binom{n+N}{n}$.
(Hint: define $\mathbb{N}^N_{=k} := \{\alpha \in \mathbb{N}^N\ :\ |\alpha| = k\}$ and observe, that $\mathbb{N}^N_{\leq n} = \cup_{k=0}^n M_{=k}$.)

5. Let X be a $\mathbb{K}-$vector space endowed with a family $\{\|\ \ \|_j\}_{j\in I}$ of norms. For $\epsilon > 0$, $x \in X$ and $\{j_1, \ldots, j_n\} \subset I$, one defines
$$U_{x,\epsilon,j_1,\ldots,j_n} := \{y \in X\ :\ \forall k = 1, \ldots, n,\ \|y - x\|_{j_k} < \epsilon\}.$$
Show that the collection of all subsets $U \subset X$, so that
$$\forall x \in U,\ \exists \epsilon > 0,\ \exists \{j_1, \ldots, j_n\} \subset I \text{ s.t. } U_{x,\epsilon,j_1,\ldots,j_n} \subset U$$
is a topology τ_X on X, i.e.:

- $\forall \mathcal{F} \subset \tau_X$, $|\mathcal{F}| \in \mathbb{N}$ implies $\cap_{U\in\mathcal{F}} U \in \tau_X$,
- $\forall \mathcal{F} \subset \tau_X$, $\cup_{U\in\mathcal{F}} U \in \tau_X$.

6. Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^N)$ be a sequence which is Cauchy for all the norms $|||\ |||_n$. Show, that this sequence converges to some $f \in \mathcal{S}(\mathbb{R}^N)$ for τ_S.
(Hint: you might wanna use the completeness of spaces like $C(K)$ or $C_0(K)$ for the uniform norm $\|\ \|_\infty$ on a compact set K and the uniform continuity of the Riemann integral.)

7. Let $g \in L^2(\mathbb{R}^N, \mu_L)$. For $x \in \mathbb{R}^N$, set $E_x := \{y \in \mathbb{R}^N\ :\ y = \delta x \text{ s.t. } \delta \in [0,1]^N\}$. Show that the function

$$\mathbb{R}^N \ni x \mapsto G(x) := \operatorname{sgn}(x) \int_{E_x} g(y)\mu_L(dy)$$

is well-defined, continuous and polynomially bounded. If $f \in \mathcal{S}(\mathbb{R}^N)$, show that

$$\int_{\mathbb{R}^N} G(x)\partial^{\bar{1}} f(x)\mu_L(dx) = (-1)^N \int_{\mathbb{R}^N} g(x)f(x)\mu_L(dx).$$

(Hint: for the second part, show it first when $g(x) = \prod_{k=1}^N g_k(x_k)$ and all $g_k(t)$ are continuous and compactly supported on $\mathbb{R}$. Use then a density argument.)

8. Find an $n \in \mathbb{N}$ and continuous and polynomially bounded functions $(g_\alpha(x))_{\alpha\in\mathbb{N}^N_{\leq n}}$ on $\mathbb{R}$, so that

$$\varphi(f) = \sum_{\alpha\in\mathbb{N}^N_{\leq n}} \int_{\mathbb{R}} g_\alpha(x)\partial^\alpha f(x)\mu_L(dx)$$

for

(a) $\varphi = \delta(x)$,
(b) $\varphi = \text{p.v.}(\frac{1}{x})$.

Can you find more than one such representations?

9. Prove that sequence $(h_k)_{k\in\mathbb{N}^*} \subset \mathcal{S}'(\mathbb{R})$ converges in the weak* topology to $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ for

(a) $\varphi = \delta(x)$ and $h_k(x) = 1_{[-1,1]}\frac{k}{2}e^{-k|x|}$,
(b) $\varphi = \text{p.v.}(\frac{1}{x})$ and $h_k(x) = \frac{x}{x^2+k^{-2}}$.

10. Prove that the weak* topology on $\mathcal{S}'(\mathbb{R}^N)$ is a topology (see exercice 5).
Prove that $\mathcal{S}'(\mathbb{R}^N)$ is complete when endowed with this topology.