

Exercice sheet 6

Symmetries in quantum physics

1. For a Hilbert space \mathcal{H} , let

$$\pi : \mathcal{H} \rightarrow \mathbb{P}(\mathcal{H}), \quad v \mapsto \pi(v) := [v].$$

Show that

$$\tau_\pi := \{U \in \mathcal{P}(\mathbb{P}(\mathcal{H})) : \pi^{-1}(U) \in \tau_{\mathcal{H}}\}$$

is a topology on $\mathbb{P}(\mathcal{H})$ and that this is the finest topology for which π is a continuous map.

Verify that π maps open sets to open sets.

Show that $(\mathbb{P}(\mathcal{H}), \tau_\pi)$ is a **Hausdorff space** (for two points $x, y \in \mathbb{P}(\mathcal{H})$, there are two disjoint sets $U, V \in \tau_\pi$, so that $U \ni x$ and $V \ni y$).

2. With the same notations as in the previous exercise and for $B \subset \mathbb{P}(\mathcal{H})$, show that $\overline{B} = p(p^{-1}\{\overline{B}\})$ and that $\overline{B} = (B^\perp)^\perp$.

For a given unit vector $e \in \mathcal{H}$, set $U_e := \mathbb{P}(\mathcal{H}) \setminus \mathbb{P}(e^\perp)$. Show that the map

$$\chi_e : U_e \rightarrow e^\perp, \quad \chi_e(x) = \frac{x - e\langle e, x \rangle}{\langle e, x \rangle}$$

is well-defined and that it is a homeomorphism from U_e onto e^\perp . Show that its inverse is $\chi_e^{-1}(x) = [e + x]$.

3. We recall here that a map $\mathcal{T} : \mathbb{P}(\mathcal{H}) \rightarrow \mathbb{P}(\mathcal{H})$ is a **morphism** if it is continuous for the quotient topology and that it is a **symmetry** if it is a bijection which preserves transition probabilities, i.e. if $\text{tr}(P_x P_y) = \text{tr}(P_{x'} P_{y'})$ for $[x'] = \mathcal{T}([x])$ and $[y'] = \mathcal{T}([y])$.

Show that for a symmetry, if $B \subset \mathbb{P}(\mathcal{H})$, $\mathcal{T}(B^\perp) \subset \mathcal{T}(B)^\perp$. Show then that, $\mathcal{T}(\overline{B}) = \overline{\mathcal{T}(B)}$. Conclude, that a symmetry is an isomorphism, i.e. a continuous bijection with continuous inverse.

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