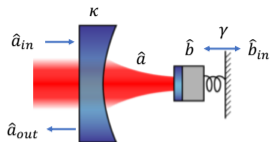




# Quantum Electrodynamics and Quantum Optics: Lecture 13

Fall 2025

# Ponderomotive squeezing<sup>1</sup>



## System Hamiltonian

$$H = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b} + \frac{\hbar\omega_c}{L} x_{\text{zpf}} \hat{a}^\dagger \hat{a} (\hat{b} + \hat{b}^\dagger)$$

Introducing position and momentum operators  $\hat{Q} = \frac{1}{2}(\hat{b} + \hat{b}^\dagger)$ ,  $\hat{P} = \frac{i}{2}(\hat{b} - \hat{b}^\dagger)$ , and coupling constant  $g_0 = \frac{\omega_c}{L} x_{\text{zpf}}$ , we get

$$H = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\Omega_m \left[ \hat{P}^2 + \left( \hat{Q} + \frac{g_0}{\Omega_m} \hat{a}^\dagger \hat{a} \right)^2 \right] - \frac{\hbar g_0^2}{\Omega_m} (\hat{a}^\dagger \hat{a})^2.$$

To isolate the light field dynamics from the mechanics, we introduce *polaron transformation*  $\hat{s} = \exp\left(i \frac{g_0}{\Omega_m} \hat{a}^\dagger \hat{a} \hat{P}\right)$  to transform our Hamiltonian using

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \dots,$$

<sup>1</sup>Quantum Optomechanics - Bowen - Chapter 4

## Ponderomotive squeezing

we get

$$\hat{s}^\dagger \hat{Q} \hat{s} = \hat{Q} - \frac{g_0}{\Omega_m} \hat{a}^\dagger \hat{a}.$$

The polaron transformation cancels the displacement caused by radiation pressure by applying an opposite but equivalent displacement via the unitary operation. The Hamiltonian would thus transform into

$$\hat{H}' = \hat{s}^\dagger \hat{H} \hat{s} = \hbar \omega_c \hat{a}^\dagger \hat{a} + \hbar \Omega_m \left( \hat{P}^2 + \hat{Q}^2 \right) - \frac{\hbar g_0^2}{\Omega_m} (\hat{a}^\dagger \hat{a})^2.$$

We can now linearize it using  $\hat{a} \rightarrow \alpha + \delta \hat{a}$ , so that

$$(\hat{a}^\dagger \hat{a})^2 \rightarrow \alpha^4 + 2\alpha^3 (\delta \hat{a}^\dagger + \delta \hat{a}) + 4\alpha^2 \delta \hat{a} \delta \hat{a}^\dagger + \alpha^2 (\delta \hat{a}^{\dagger 2} + \delta \hat{a}^2)$$

$$\frac{g_0^2}{\Omega_m} \alpha^2 (\delta \hat{a}^{\dagger 2} + \delta \hat{a}^2) \Leftrightarrow S(\zeta) = \exp \left( \frac{1}{2} \zeta^* \hat{a}^2 + \frac{1}{2} \zeta \hat{a}^{\dagger 2} \right),$$

whose unitary evolution leads to a squeezing operator.

## Quantum Langevin Equations

We first move to an interaction frame where  $\omega_c$  is replaced by  $\Delta = \omega_c - \omega_L$  where  $\omega_L$  is the laser driving frequency. To include the dissipation and noise from the environment, we derive

### Quantum Langevin equations

$$\begin{aligned}\dot{\hat{a}}(t) &= - \left( i\Delta + \frac{\kappa}{2} \right) \hat{a} - ig_0 \hat{a} (\hat{b} + \hat{b}^\dagger) + \sqrt{\kappa} \hat{a}_{\text{in}}(t), \\ \dot{\hat{b}}(t) &= - \left( i\Omega_m + \frac{\Gamma_m}{2} \right) \hat{b} - ig_0 \hat{a}^\dagger \hat{a} + \sqrt{\Gamma_m} \hat{b}_{\text{in}}(t),\end{aligned}$$

where  $\Gamma_m$  is the mechanical dissipation rate. Here we assume  $g_0 \ll \kappa$  in order to linearize the equations  $\hat{a} \rightarrow \alpha + \hat{a}$ . Introducing  $g = g_0 \sqrt{\alpha}$  and by going to the Fourier domain,

$$\begin{aligned}\hat{a}[\omega] &= \frac{\sqrt{\kappa} \hat{a}_{\text{in}} - ig(\hat{b}[\omega] + \hat{b}^\dagger[\omega])}{i(\Delta - \omega) + \kappa/2}, \\ \hat{b}[\omega] &= \frac{\sqrt{\Gamma_m} \hat{b}_{\text{in}}[\omega]}{i(\Omega_m - \omega) + \Gamma_m/2} - \frac{ig(\hat{a}[\omega] + \hat{a}^\dagger[\omega])}{i(\Omega_m - \omega) + \Gamma_m/2}.\end{aligned}$$

## Quantum Langevin Equations

The mechanical motion can be expressed as a response to the environmental noise and optical input fluctuations

$$\hat{b}[\omega] = \frac{\sqrt{\Gamma_m} \hat{b}_{\text{in}}[\omega]}{i(\Omega'_m - \omega) + \Gamma'/2} + \frac{ig}{i(\Delta - \omega) + \kappa/2} \frac{-\sqrt{\kappa} \hat{a}_{\text{in}}[\omega]}{i(\Omega'_m - \omega) + \Gamma'/2} \\ + \frac{ig}{-i(\Delta - \omega) + \kappa/2} \frac{-\sqrt{\kappa} \hat{a}_{\text{in}}^\dagger[\omega]}{i(\Omega'_m - \omega) + \Gamma'/2}$$

### Optical spring and dynamical back-action

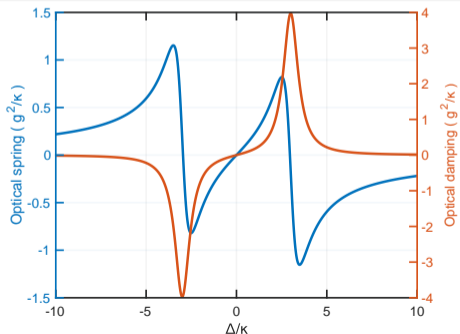
where  $\Omega'_m = \Omega_m + \delta\Omega_m$  and  $\Gamma' = \Gamma_m + \delta\Gamma_m$  with

$$\delta\Omega_m = g^2 \text{Im} \left[ \frac{1}{i(\Delta - \Omega_m) + \kappa/2} - \frac{1}{-i(\Delta + \Omega_m) + \kappa/2} \right] \\ \delta\Gamma_m = 2g^2 \text{Re} \left[ \frac{1}{i(\Delta - \Omega_m) + \kappa/2} - \frac{1}{-i(\Delta + \Omega_m) + \kappa/2} \right].$$

# Optical spring and dynamical back-action

## Optical spring and dynamical back-action

$$\delta\Omega_m = g^2 \text{Im} \left[ \frac{1}{i(\Delta - \Omega_m) + \kappa/2} - \frac{1}{-i(\Delta + \Omega_m) + \kappa/2} \right]$$
$$\delta\Gamma_m = 2g^2 \text{Re} \left[ \frac{1}{i(\Delta - \Omega_m) + \kappa/2} - \frac{1}{-i(\Delta + \Omega_m) + \kappa/2} \right].$$



## Quasi-static approximation<sup>2</sup>

Here we make a few approximations to simplify the derivation:

### Quasi-static approximation

- 1  $\Delta = 0$ : the laser is tuned exactly to the optical cavity resonance frequency.
- 2  $\kappa \gg \Omega_m$ : Bad cavity limit.
- 3  $\omega \ll \Omega_m$ : We are only interested in the quasi-static response, so the resonant response of the mechanical resonator does not play a role.

Under these assumptions together with the input output theorem  $\hat{a}_{\text{out}} + \hat{a}_{\text{in}} = \sqrt{\kappa}\hat{a}$ ,

$$i\Omega_m\hat{b}[\omega] = \sqrt{\Gamma_m}\hat{b}_{\text{in}}[\omega] - \frac{2ig}{\sqrt{\kappa}}(\hat{a}_{\text{in}}[\omega] + \hat{a}_{\text{in}}^\dagger[\omega])$$

$$\hat{a}_{\text{out}}[\omega] = \hat{a}_{\text{in}}[\omega] - \frac{2ig}{\sqrt{\kappa}}(\hat{b}[\omega] + \hat{b}^\dagger[\omega]).$$

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<sup>2</sup>Safavi-Naeini, Amir H., et al. "Squeezed light from a silicon micromechanical resonator." Nature 500.7461 (2013): 185.

## Quasi-static approximation

Define measurement rate  $\Gamma_{\text{meas}} \equiv 4g^2/\kappa$ , we can rewrite

$$\hat{a}_{\text{out}}[\omega] = \hat{a}_{\text{in}} + \frac{2i\Gamma_{\text{meas}}}{\Omega_{\text{m}}} (\hat{a}_{\text{in}}[\omega] + \hat{a}_{\text{in}}^{\dagger}[\omega]) + \frac{\sqrt{\Gamma_{\text{m}}\gamma_{\text{meas}}}}{\Omega_{\text{m}}} (\hat{b}_{\text{in}}[\omega] + \hat{b}_{\text{in}}^{\dagger}[\omega]).$$

Ignoring thermal noise  $\Gamma_{\text{m}} = 0$ , and dropping the terms of order  $(\Gamma_{\text{meas}}/\Omega_{\text{m}})^2$ , we can calculate the output optical quadrature power spectral density

### Output optical quadrature

$$S_{\hat{X}_{\theta}\hat{X}_{\theta}}^{\text{out}} = \int_{-\infty}^{+\infty} d\omega' \langle \hat{X}_{\theta}^{\text{out}}[\omega] \hat{X}_{\theta}^{\text{out}}[\omega'] \rangle = 1 + \frac{4\Gamma_{\text{meas}}}{\Omega_{\text{m}}} \sin 2\theta,$$

where the quadrature operator is defined as  $\hat{X}_{\theta} = \hat{a}e^{-i\theta} + \hat{a}^{\dagger}e^{i\theta}$ . For  $\theta = -\pi/4$  we achieve the maximum squeezing with a noise floor strongly depends on the ratio  $\Gamma_{\text{meas}}/\Omega_{\text{m}}$ .

## Quasi-static approximation

To include the effects from thermal noises, we assume the form of the correlators to be

### Thermal correlators

$$\begin{aligned}\langle \hat{b}_{\text{in}}[\omega] \hat{b}_{\text{in}}^\dagger[\omega'] \rangle &= (\bar{n}_{\text{th}}[\omega] + 1) \delta(\omega + \omega'), \\ \langle \hat{b}_{\text{in}}^\dagger[\omega] \hat{b}_{\text{in}}[\omega'] \rangle &= \bar{n}_{\text{th}}[\omega] \delta(\omega + \omega'),\end{aligned}$$

which leads to

$$S_{X_\theta X_\theta}^{\text{out}} = 1 + \frac{4\Gamma_{\text{meas}}}{\Omega_{\text{m}}} \sin 2\theta + \frac{4\Gamma_{\text{meas}}}{\Omega_{\text{m}}} \frac{\bar{n}_{\text{th}}[\omega]}{Q_{\text{m}}} (1 - \cos 2\theta).$$

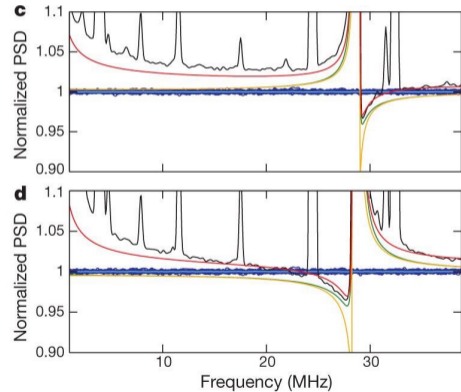
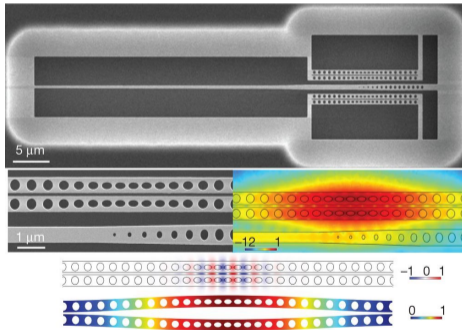
In this model there is no squeezing at  $\theta = -\pi/4$  and frequency  $\omega$  if  $\bar{n}_{\text{th}}[\omega] > Q_{\text{m}}$ , where  $Q_{\text{m}}$  is the mechanical quality factor. However some squeezing is always present, but is shifted to other quadratures and the amount of detectable squeezing is reduced at higher temperatures.

## LETTER

doi:10.1038/nature12307

### Squeezed light from a silicon micromechanical resonator

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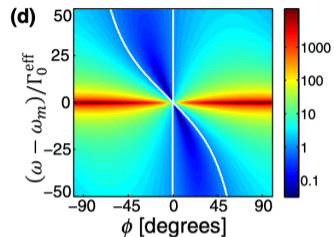
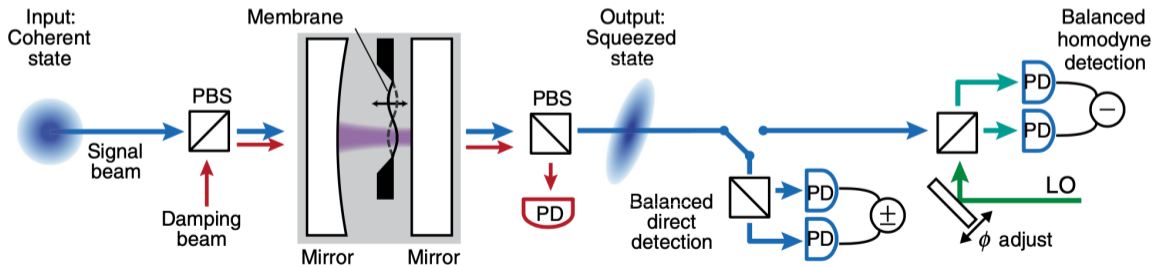


<sup>3</sup>Safavi-Naeini, Amir H., et al. "Squeezed light from a silicon micromechanical resonator." *Nature* 500.7461 (2013): 185.

## Strong Optomechanical Squeezing of Light

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<sup>4</sup>Purdy, Thomas P., et al. "Strong optomechanical squeezing of light." *Physical Review X* 3.3 (2013): 031012.

## Dynamics of mechanical resonator <sup>5</sup>

Taking into account the dynamics of the mechanical resonator while keeping the bad-cavity limit and resonant probing, we can derive the detected spectral density detected by the homodyne as

### Homodyne signal

$$S_{II}^{\theta}[\omega] = \frac{1}{2} + 8\Gamma_m^2 |\chi[\omega]|^2 n_{\text{QBA}} \left( n_{\text{th}} + n_{\text{QBA}} + \frac{1}{2} \right) \sin^2 \theta \\ + 2\Gamma_m \text{Re}[\chi[\omega]] n_{\text{QBA}} \sin 2\theta$$

where  $\chi[\omega] = \frac{\Omega_m \Gamma_m}{(\Omega_m^2 - \omega^2 - i\Gamma_m \omega)}$  is the mechanical susceptibility. Here, we introduced the effective optomechanical cooperativity  $C_{\text{eff}}[\omega] \equiv \frac{C}{(1 - 2i\omega/\kappa)^2}$  and  $n_{\text{QBA}} = |C_{\text{eff}}|$ , with  $C \equiv \frac{4g^2}{\kappa\Gamma}$  being the optomechanical cooperativity.

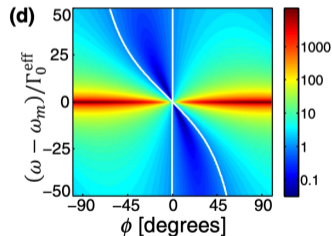
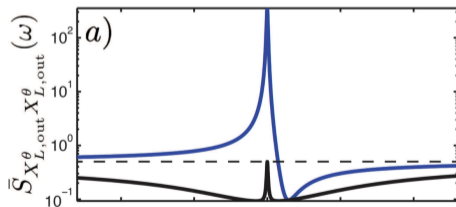
<sup>5</sup>Quantum Optomechanics - Bowen - Chapter 3,4

# Ponderomotive squeezing

## Homodyne signal

$$S_{II}^{\theta}[\omega] = \frac{1}{2} + 8\Gamma_m^2 |\chi[\omega]|^2 n_{\text{QBA}} (n_{\text{th}} + n_{\text{QBA}} + \frac{1}{2}) \sin^2 \theta + 2\Gamma_m \text{Re}[\chi[\omega]] n_{\text{QBA}} \sin 2\theta$$

Squeezing when the PSDs of different quadrature angles are probed<sup>67</sup>.



<sup>6</sup>Quantum Optomechanics - Bowen - Chapter 4

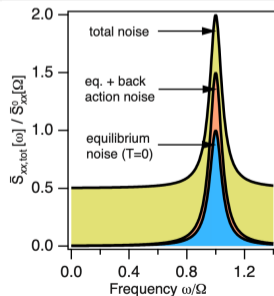
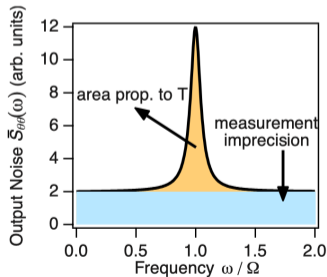
<sup>7</sup>Purdy, Thomas P., et al. "Strong optomechanical squeezing of light." *Physical Review X* 3.3 (2013): 031012.

## Standard quantum limit<sup>8</sup>

The phase quadrature ( $\theta = \frac{\pi}{2}$ ) homodyne power spectral density is simply

### Phase quadrature

$$\begin{aligned} S_{\text{phase}}[\omega] &= \frac{1}{2} + 8\Gamma_m |C_{\text{eff}}|^2 |\chi[\omega]|^2 + 4\Gamma_m |C_{\text{eff}}| S_{Q^0 Q^0}[\omega] \\ &= \frac{1}{2} + 4\Gamma_m |C_{\text{eff}}| S_{QQ}[\omega] \end{aligned}$$

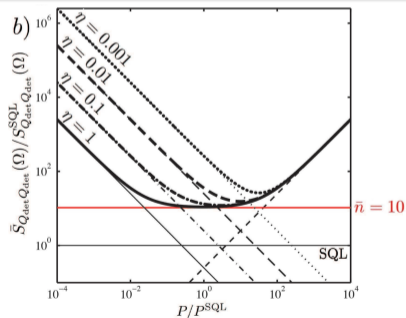
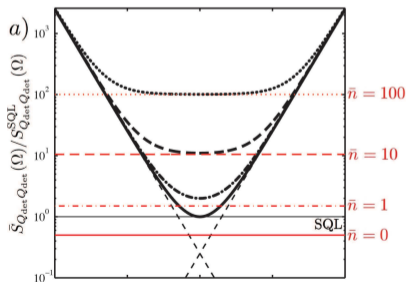


<sup>8</sup>Clerk, Aashish A., et al. "Introduction to quantum noise, measurement, and amplification." *Reviews of Modern Physics* 82.2 (2010): 1155.

# Standard quantum limit<sup>9</sup>

## Phase quadrature

$$\begin{aligned} S_{\text{phase}}[\omega] &= \frac{1}{2} + 8\Gamma_m |C_{\text{eff}}|^2 |\chi[\omega]|^2 + 4\Gamma_m |C_{\text{eff}}| S_{Q^0 Q^0}[\omega] \\ &= \frac{1}{2} + 4\Gamma_m |C_{\text{eff}}| S_{QQ}[\omega] \end{aligned}$$



<sup>9</sup>Quantum Optomechanics - Bowen - Chapter 3

# Back-action-evading measurements of nanomechanical motion

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