Quantum Electrodynamics and Quantum Optics: Lecture 10

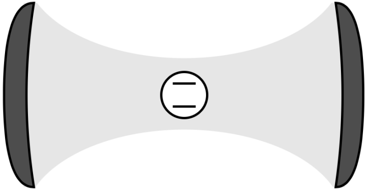
Fall 2025

Coupling an atom to a cavity: Jaynes-Cummings model

A two level system coupled to a cavity (harmonic oscillator)

Jaynes-Cummings Hamiltonian

$$\hat{H} = \underbrace{\frac{\hbar\omega_{\mathrm{eg}}}{2}\hat{\sigma}_z}_{\text{atom}} + \underbrace{\hbar\omega\hat{a}^\dagger\hat{a}}_{\text{cavity}} + \underbrace{\hbar g(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^\dagger)}_{\text{coupling}}$$



In a subspace span by the states $|g, n+1\rangle$ and $|e, n\rangle$ the Hamiltonian is given by:

$$\hat{H}_n = \begin{bmatrix} n\hbar\omega + \frac{\hbar\omega_{\mathrm{eg}}}{2} & \hbar g\sqrt{n+1} \\ \hbar g\sqrt{n+1} & (n+1)\hbar\omega - \frac{\hbar\omega_{\mathrm{eg}}}{2} \end{bmatrix}$$

Note: The off-diagonal terms are equal

$$\langle g, n+1|\,\hat{\sigma}^-\hat{a}^\dagger\,|e, n\rangle = \sqrt{n+1}$$
$$\langle e, n|\,\hat{\sigma}^+\hat{a}\,|g, n+1\rangle = \sqrt{n+1}$$

Diagonalizing Jaynes-Cummings Hamiltonian[1,2]

The Hamiltonian can be diagonalized by the following rotation in the $\{|g,n+1\rangle, |e,n\rangle\}$ subspace. Introducing atom-cavity detuning $\Delta = \omega - \omega_{\text{eg}}$

Eigenstates of the Jaynes-Cummings Hamiltonian

$$
\begin{aligned}
|+\rangle_n &= \underbrace{\frac{2g\sqrt{n+1}}{\sqrt{(\Omega_n-\Delta)^2+4g^2(n+1)}}}_{\cos\theta_n} |e,n\rangle - \underbrace{\frac{\Omega_n-\Delta}{\sqrt{(\Omega_n-\Delta)^2+4g^2(n+1)}}}_{\sin\theta_n} |g,n+1\rangle \\
|-\rangle_n &= \sin\theta_n |e,n\rangle + \cos\theta_n |g,n+1\rangle
\end{aligned}
$$

where $\theta_n = \frac{1}{2}\tan^{-1}\left(\frac{g\sqrt{n+1}}{\Delta}\right)$ and $\Omega_n \equiv \sqrt{4g^2(n+1)+\Delta^2}$ is the multiphoton off-resonant Rabi frequency. Recall: $\Omega_{\text{Rabi}} = 2g = \Omega_0(\Delta=0)$ - vacuum Rabi splitting.

---

[1] Quantum Optics, Scully, chapter 19.3

[2] Homework 10 Exercise 1

Two regimes of the Jaynes-Cummings interaction

An important approximation $\Delta \gg g(n+1)$

$$\Omega_n - \Delta = \left(\Delta \sqrt{1 + \frac{4g^2(n+1)}{\Delta^2}} - \Delta \right) \approx \frac{2g^2(n+1)}{\Delta}$$

Resonant case $\Delta = 0$

Eigenstates are fully entangled states

$$|+\rangle_n = \frac{1}{\sqrt{2}}(|e,n\rangle - |g,n+1\rangle)$$
$$|-\rangle_n = \frac{1}{\sqrt{2}}(|e,n\rangle + |g,n+1\rangle)$$

Dispersive regime $\Delta \gg g(n+1)$

Eigenstates approximate the uncoupled case

$$|+\rangle_n \approx |e,n\rangle - \frac{g\sqrt{n+1}}{\Delta} |g,n+1\rangle$$
$$|-\rangle_n \approx |g,n+1\rangle + \frac{g\sqrt{n+1}}{\Delta} |e,n\rangle$$

Two regimes of the Jaynes-Cummings interaction

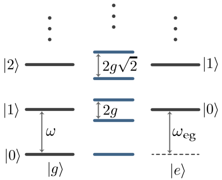
The exact eigenenergies of the Jaynes-Cumings model

$$
\begin{aligned}
E_{\mathrm{n}+} &= \hbar\omega n \qquad\qquad + \frac{\hbar}{2}\omega_{\mathrm{eg}} - \frac{\hbar}{2}(\Omega_n - \Delta) \\
E_{\mathrm{n}-} &= \hbar\omega(n+1) - \frac{\hbar}{2}\omega_{\mathrm{eg}} + \frac{\hbar}{2}(\Omega_n - \Delta)
\end{aligned}
$$

Resonant case $\Delta = 0$

Photon number dependent Rabi splitting

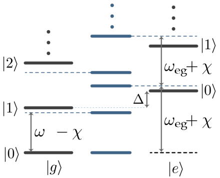
$$\frac{\Delta E_n}{\hbar} = 2g\sqrt{n+1}$$



Dispersive regime $\Delta \gg g(n+1)$

Dispersive shift $\chi = g^2/\Delta$

$$
\begin{aligned}
\frac{E_{\mathrm{n}+}}{\hbar} &= \omega n \qquad\qquad + \frac{\omega_{\mathrm{eg}}}{2} - \chi(n+1) \\
\frac{E_{\mathrm{n}-}}{\hbar} &= \omega(n+1) - \frac{\omega_{\mathrm{eg}}}{2} + \chi(n+1)
\end{aligned}
$$



Reverse engineering the Hamiltonian of the dispersive regime of cQED

Dispersive-limit ($\Delta \gg 2g\sqrt{n+1}$) eigenstates and eigenenergies

$$|+\rangle_n \approx |e,n\rangle \qquad E_{+n} = \hbar\omega n + \frac{\hbar\omega_{eg}}{2} - \frac{\hbar g^2}{\Delta} n - \frac{\hbar g^2}{\Delta}$$

$$|-\rangle_n \approx |g,n+1\rangle \qquad E_{-n} = \hbar\omega(n+1) - \frac{\hbar\omega_{eg}}{2} \underbrace{+}_{\hat{\sigma}_z} \frac{\hbar g^2}{\Delta} \underbrace{(n+1)}_{\hat{a}^\dagger\hat{a}} \underbrace{}_{\text{Lamb shift}}$$

Knowing the approximate eigenstates and the eigenenergies of the system in the dispersive limit we can guess the form of the Hamiltonian in terms of operators $\hat{a}, \hat{a}^\dagger$ and $\hat{\sigma}_z$:

Dispersive Hamiltonian

$$\hat{H}_{\text{disp}} = \underbrace{\hbar\omega\hat{a}^\dagger\hat{a}}_{\text{cavity}} + \underbrace{\frac{\hbar\omega_{eg}}{2}\hat{\sigma}_z}_{\text{atom}} + \underbrace{\hbar\chi(\hat{a}^\dagger\hat{a} + \frac{1}{2})\hat{\sigma}_z}_{\text{dispersive interaction}}, \text{ where } \chi = \frac{g^2}{\Delta}$$

Schrieffer-Wolff (Polaron) Transformation[3]

Equivalently the dispersive Hamiltonian can be obtained by making the following unitary transformation and keeping only the leading terms in g/Δ:

$$\hat{U} = \exp\left[\frac{g}{\Delta}(\hat{a}\hat{\sigma}^{+} - \hat{a}^{\dagger}\hat{\sigma}^{-})\right]$$

Expanding up to the second order in g/Δ we recover again the dispersive Hamiltonian:

$$\hat{U}\hat{H}\hat{U}^{\dagger} \approx \hbar\left[\omega + \frac{g^2}{\Delta}\hat{\sigma}_z\right]\hat{a}^{\dagger}\hat{a} + \frac{\hbar}{2}\left[\omega_{\mathrm{eg}} + \frac{g^2}{\Delta}\right]\hat{\sigma}_z$$

[3] Blais, Alexandre, et al. "Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation." Physical Review A 69.6 (2004): 062320.

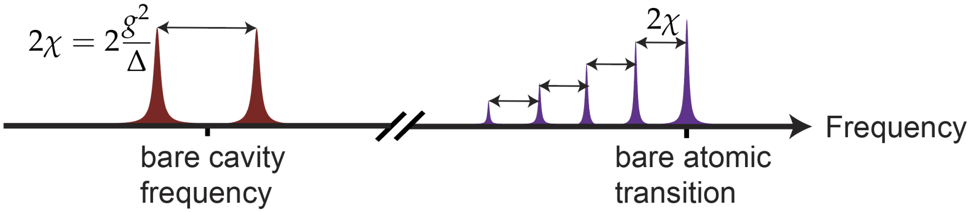
Two interpretations of the dispersive interaction

Qubit state dependent resonator pulling

$$\hat{H}_{\text{disp}} = \underbrace{\hbar\left(\omega + \chi\hat{\sigma}_z\right)\hat{a}^\dagger\hat{a}}_{\text{atom-dependent cavity}} + \underbrace{\frac{\hbar}{2}\left(\omega_{\text{eg}} + \chi\right)\hat{\sigma}_z}_{\text{Lamb shifted atom}}$$

AC-Stark shift

$$\hat{H}_{\text{disp}} = \underbrace{\hbar\omega\hat{a}^\dagger\hat{a}}_{\text{cavity}} + \underbrace{\frac{\hbar}{2}\left(\omega_{\text{eg}} + 2\chi(\hat{a}^\dagger\hat{a} + \frac{1}{2})\right)\hat{\sigma}_z}_{\text{photon number dependent shift}}$$



As is clear from this expression, the atom transition is shifted by $\chi(n + 1/2)$. Alternatively, one can interpret the shift as a dispersive shift of the cavity transition by $\chi\hat{\sigma}_z$. In other words, the atom pulls the cavity frequency by $\pm\chi$.

Dispersive interaction as a quantum non-demolition (QND) measurement

Dispersive Hamiltonian

$$\hat{H}_{\text{disp}} = \hbar\omega\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_{\text{eg}}}{2}\hat{\sigma}_z + \hbar\chi(\hat{a}^\dagger\hat{a} + \frac{1}{2})\hat{\sigma}_z,$$

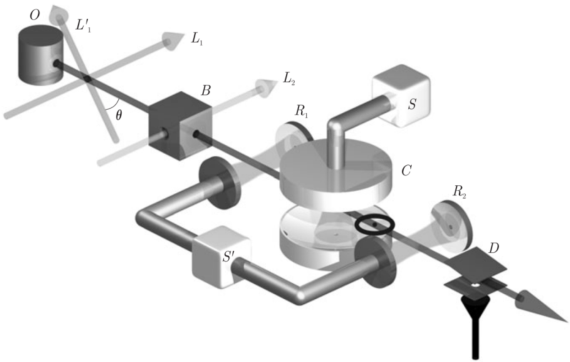
Note that $|n, e\rangle$ and $|n+1, g\rangle$ are both eigenstates of $\hat{H}_{\text{disp}}$, i.e. the commutators of $\hat{\sigma}_z$ and $\hat{a}^\dagger\hat{a}$ with $\hat{H}_{\text{disp}}$ vanish.

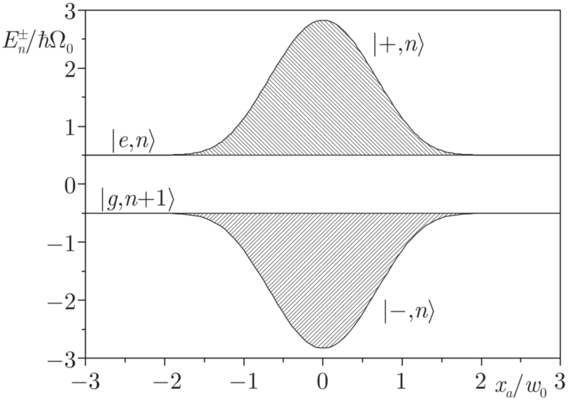
$$\left[\hat{H}_{\text{disp}}, \hat{\sigma}_z\right] = 0, \quad \left[\hat{H}_{\text{disp}}, \hat{a}^\dagger\hat{a}\right] = 0$$

Thus upon a projective measurement of either the qubit state or the photon number in the cavity the cQED system in the dispersive regime remains in this state. This constitutes a *quantum non-demolition measurement*. At the same time

$$\left[\hat{H}_{\text{disp}}, \hat{\sigma}_x\right] \neq 0 \text{ – measurement of } \hat{\sigma}_x \text{ is not QND}$$

Atom passing through a cavity[1]

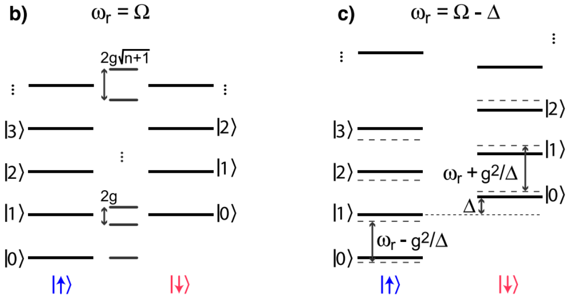




Position of the dressed energies $E_{\pm\mathrm{n}}$, in units of $\hbar\Omega_0$, as a function of the atomic position x_a, in units of the cavity mode waist w_0, for $\Delta_c = \Omega_0$ and $n = 30$. The energy origin is taken as $(n + 1/2)\hbar\omega_c$.

[1] Haroche, Serge, and J-M. Raimond. Exploring the quantum: atoms, cavities, and photons. Oxford university press, 2006.

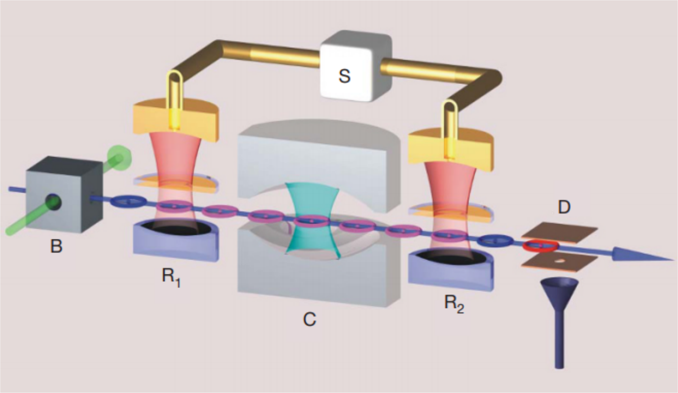
Dispersive regime of cavity QED: Level diagram [3]



Energy spectrum of the uncoupled (left and right) and dressed (center) atom-photon states in the case of zero detuning. The degeneracy of the two-dimensional manifolds of states with $n-1$ quanta is lifted by $2g\sqrt{n+1}$. c) Energy spectrum in the dispersive regime (long dash lines). To second order in g, the level separation is independent of n, but depends on the state of the atom.

[3] Blais, Alexandre, et al. "Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation." Physical Review A 69.6 (2004): 062320.

Example: Birth, life and death of a photon[4]



A thermal beam of rubidium atoms passing through a superconducting microwave cavity in the dispersive limit.

[4] Gleyzes, Sebastien, et al. "Quantum jumps of light recording the birth and death of a photon in a cavity." Nature 446.7133 (2007): 297.

Example: Birth, life and death of a photon[4]

When cavity C contains n photons, the uncoupled atom-cavity states $|e, n\rangle$ and $|g, n\rangle$ evolve into dressed states, shifted respectively, in angular frequency unit, by: $+(\sqrt{\Delta_c + (n+1)g^2} - \Delta_c)/2$ and $-(\sqrt{\Delta_c + ng^2} - \Delta_c)/2$. The difference between these frequencies integrated overtime, yields the phase shift of atom passing the cavity $\Phi(n, \Delta_c)$.

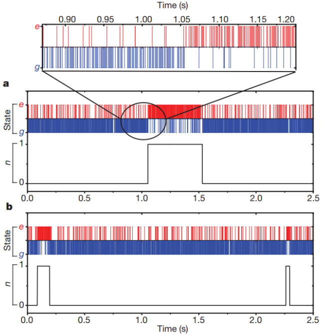
π phase shift per photon

$$\Phi(1, \Delta_c) - \Phi(0, \Delta_c) = \pi$$

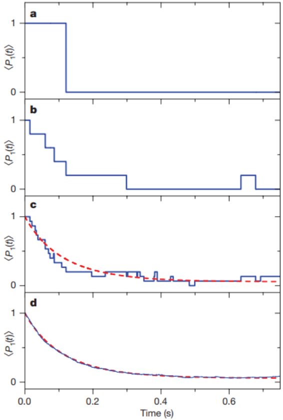
If this condition is met, the Ramsey pulse in R_2 ideally brings the atom into g if $n = 0$ and e if $n = 1$.

[4] Gleyzes, Sebastien, et al. "Quantum jumps of light recording the birth and death of a photon in a cavity." Nature 446.7133 (2007): 297.

Example: Birth, life and death of a photon[4]



QND detection of a single photon (Quantum jumps).



Measurement of the decay of a photon.

[4] Gleyzes, Sebastien, et al. "Quantum jumps of light recording the birth and death of a photon in a cavity." Nature 446.7133 (2007): 297.

Paper for next week's presentation

Reconstruction of non-classical cavity field states with snapshots of their decoherence

Samuel Deléglise[1], Igor Dotsenko[1,2], Clément Sayrin[1], Julien Bernu[1], Michel Brune[1], Jean-Michel Raimond[1] & Serge Haroche[1,2]

Questions for next week's paper

- What are the different time scales in the system, e.g. photon life time, atom life time, atom-photon interaction time?
- How many thermal photons are there in the cavity due to the finite temperature of the environment?
- What is the initial state prepared in the cavity?
- How are the atoms and their states used in this experiment? How does the energy level separation compare to the cavity frequency?
- How is the atomic Ramsey interferometer used to obtain the cavity photon number?