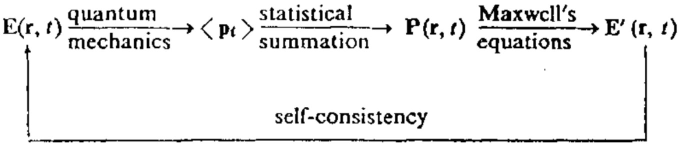
Quantum Electrodynamics and Quantum Optics: Lecture 8

Fall 2025

Semiclassical atom light interaction: Maxwell-Schrödinger equations [1]



Atomic polarization : bridging quantum mechanical and classical descriptions

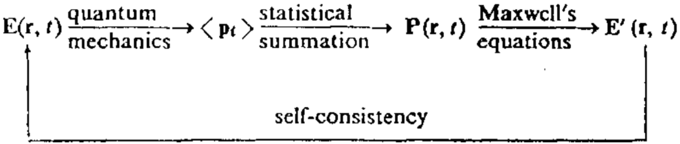
The medium is described by its susceptibility $\chi(\tau)$

$$\mathbf{P}(z,t) = \varepsilon_0 \int_{-\infty}^{t} \chi(t-\tau)\mathbf{E}(z,\tau)d\tau$$

On the other hand $\mathbf{P}(z,t) = N\langle\hat{\mathbf{p}}\rangle$ where $\hat{\mathbf{p}} = q\hat{\mathbf{r}}$ and N is the density of dipoles. For a state $|\psi\rangle = c_1|1\rangle + c_2|2\rangle$, we denote the dipole matrix element $\langle 1| q\hat{\mathbf{r}} |2\rangle = \mathbf{p}_{12} = \mathbf{p}_{21}^*$. Note that $\hat{\mathbf{r}}$ is actually a measure of distance (not position), hence the diagonal elements are $\mathbf{p}_{ii} = 0$.

[1] Laser physics; Sargent, Scully, Lamb

Semiclassical atom light interaction: Maxwell-Schrödinger equations [2]



Atomic polarization : bridging quantum mechanical and classical descriptions

We compute $\hat{\mathbf{p}} = q\hat{\mathbf{r}}$ using $|\psi\rangle = c_1 |1\rangle + c_2 |2\rangle$ and $\langle 1| q\hat{\mathbf{r}} |2\rangle = \mathbf{p}_{12} = \mathbf{p}_{21}^*$,

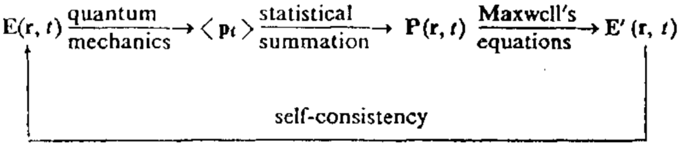
$$\langle q\hat{\mathbf{r}}\rangle = \mathrm{Tr}\{\rho\hat{\mathbf{p}}\} = \langle\psi| \hat{\mathbf{p}} |\psi\rangle = c_1^* c_2 \mathbf{p}_{12} + c_2^* c_1 \mathbf{p}_{21} = \rho_{21}\mathbf{p}_{12} + \rho_{12}\mathbf{p}_{21}$$

with density matrix

$$\hat{\rho} = |\psi\rangle\langle\psi| = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} |c_1|^2 & c_1 c_2^* \\ c_1^* c_2 & |c_2|^2 \end{pmatrix}$$

[2] Laser physics; Sargent, Scully, Lamb

Semiclassical atom light interaction: Maxwell-Schrödinger equations [3]



Helmholtz equation : semi-classical evolution

Propagation of electromagnetic waves in a medium is governed by the Helmholtz equation:

$$\underbrace{\frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}}_{\text{Homogeneous part describing plane waves}} = \underbrace{\mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}}_{\text{Interaction with the medium}}$$

The quantum mechanical part enters in the polarization.

[3] Laser physics; Sargent, Scully, Lamb

Evolution of density matrix

Von Neumann equation

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}]$$

To derive a realistic refractive index and susceptibility, we need to account for the physical dissipation. This dissipation can be introduced with a spontaneous emission model, which at this stage corresponds to the addition of an "ad-hoc" decay of the density matrix (formally through the master equation).

$$\Gamma_{12} = A_{12} = \frac{1}{4\pi\varepsilon_0}\frac{4\omega^3|\mathbf{p}_{12}|}{3\hbar c^3}$$

is the rate of spontaneous emission (derived later in the lecture) and we have the

Master equation in Linblad form

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \Gamma_{12}\left(\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \frac{1}{2}\left\{\hat{\rho}, \hat{\sigma}_+\hat{\sigma}_-\right\}\right)$$

Evolution of density matrix

Projecting the master equation onto the states $|1\rangle, |2\rangle$, with $\hat{H} = \frac{\hbar\Delta}{2}\hat{\sigma}_z + q\hat{\mathbf{r}} \cdot \mathbf{E}$ in the rotating frame with detuning $\Delta = \omega_{12} - \omega$, we obtain the

Optical Bloch equations

$$\frac{\mathrm{d}\rho_{11}}{\mathrm{d}t} = -\frac{i}{\hbar}(\mathbf{p}_{12} \cdot \mathbf{E})\rho_{12} + \text{c.c.} + \Gamma_{12}\rho_{22}$$

$$\frac{\mathrm{d}\rho_{22}}{\mathrm{d}t} = +\frac{i}{\hbar}(\mathbf{p}_{12} \cdot \mathbf{E})\rho_{12} + \text{c.c.} - \Gamma_{12}\rho_{22} = -\frac{\mathrm{d}\rho_{11}}{\mathrm{d}t}$$

$$\frac{\mathrm{d}\rho_{12}}{\mathrm{d}t} = -i\Delta\rho_{12} + \frac{i}{\hbar}(\mathbf{p}_{12} \cdot \mathbf{E})(\rho_{11} - \rho_{22}) - \frac{\Gamma_{12}}{2}\rho_{12}$$

We can extract the polarization in the steady-state $\dot{\rho}_{ij} = 0$ from $\rho_{12}(\omega)$ (recall $\mathbf{p}_{12} = \langle 1| q\hat{\mathbf{r}} |2\rangle = -\langle 1| e\hat{\mathbf{r}} |2\rangle$)

$$\rho_{12}(\omega) = \frac{i}{\hbar}\frac{\mathbf{p}_{12} \cdot \mathbf{E}}{\Gamma_{12} + i\Delta}(\rho_{11} - \rho_{22})$$

Evolution of density matrix

In the Fourier domain, using $\mathbf{p}_{21} = \mathbf{p}_{12}^*$ and for N atoms,

$$\mathbf{P}(\omega) = \varepsilon_0 \chi(\omega) \mathbf{E} = N \langle \hat{\mathbf{p}} \rangle = N \mathbf{p}_{12}^* \rho_{12}(\omega) + \text{c.c.} = \frac{i}{\hbar} \frac{N |\mathbf{p}_{12}|^2 \mathbf{E}}{\Gamma_{12} + i\Delta} (\rho_{11} - \rho_{22})$$

thus we obtain for the susceptibility $\chi = \chi'_{\text{Re}} + i\chi''_{\text{Im}}$

$$\chi'_{\text{Re}} = N \frac{|\mathbf{p}_{12}|^2}{\varepsilon_0 \hbar} \frac{\Delta}{\Gamma_{12}^2 + \Delta^2} (\rho_{11} - \rho_{22})$$

$$\chi''_{\text{Im}} = N \frac{|\mathbf{p}_{12}|^2}{\varepsilon_0 \hbar} \frac{\Gamma_{12}}{\Gamma_{12}^2 + \Delta^2} (\rho_{11} - \rho_{22})$$

Semiclassical model : derivation[4]

Plane wave solution

Considering light linearly polarized along $\boldsymbol{\epsilon}$ and introducing slowly varying amplitudes $\mathcal{E}(z,t)$ and $\mathcal{P}(z,t)$ with phase $\phi(z,t)$

$$\mathbf{E}(z,t) = \boldsymbol{\epsilon}\mathcal{E}(z,t)e^{-i(\omega t - kz + \phi(z,t))} + \text{c.c.} = \boldsymbol{\epsilon}E(z,t)$$

$$\mathbf{P}(z,t) = \boldsymbol{\epsilon}\mathcal{P}(z,t)e^{-i(\omega t - kz + \phi(z,t))} + \text{c.c.} = \boldsymbol{\epsilon}P(z,t)$$

$$\mathcal{P}(z,t) = \varepsilon_0\mathcal{E}\chi(\omega) = \varepsilon_0\mathcal{E}\left(\chi'_{\text{Re}} + i\chi''_{\text{Im}}\right)$$

We assume slowly varying amplitudes and phase, i.e.

$$\begin{cases} \frac{\partial\mathcal{E}}{\partial t} \ll \omega\mathcal{E}, & \frac{\partial\mathcal{P}}{\partial t} \ll \omega\mathcal{P}, & \frac{\partial\phi}{\partial t} \ll \omega \\ \frac{\partial\mathcal{E}}{\partial z} \ll k\mathcal{E}, & \frac{\partial\mathcal{P}}{\partial z} \ll k\mathcal{P}, & \frac{\partial\phi}{\partial z} \ll k \end{cases} \tag{1}$$

We will derive the corrections to the linear dispersion relation $\omega \simeq ck$

[4] Scully, M.O., Zubairy, M.S. "Quantum optics" (1999). Chapter 5, Section 4

Semiclassical model : derivation

Consider the Helmholtz equation

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\left(-\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\mathbf{E}=-\mu_0\frac{\partial^2}{\partial t^2}\mathbf{P}(z,t)$$

Applying the slowly varying envelope approximation on the first part with $\omega = ck$ yields :

$$\left(-\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)E\simeq -2ikE+\cancel{\left(-\frac{\partial \mathcal{E}}{\partial z}+i\mathcal{E}\frac{\partial \phi}{\partial z}+\frac{1}{c}\frac{\partial \mathcal{E}}{\partial t}-\frac{i\mathcal{E}}{c}\frac{\partial \phi}{\partial t}\right)e^{-i(\omega t-kz+\phi(z,t))}}$$

thus the total left-hand side is

$$-2ik\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)E=-2ik\left(\frac{\partial \mathcal{E}}{\partial z}+\frac{1}{c}\frac{\partial \mathcal{E}}{\partial t}+i\left(k-\frac{\omega}{c}-\frac{\partial \phi}{\partial z}-\frac{1}{c}\frac{\partial \phi}{\partial t}\right)\mathcal{E}\right)e^{-i(\omega t-kz+\phi(z,t))}$$

Semiclassical model : derivation

Applying the same slowly varying envelope approximation for the right-hand side of Helmholtz equation yields

$$
\begin{aligned}
-\mu_0 \frac{\partial^2 P}{\partial t^2} &= -\frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\left(\left(-i\omega\mathcal{P} + \cancel{\frac{\partial \mathcal{P}}{\partial t} - \frac{\partial \phi}{\partial t}}\mathcal{P}\right)e^{-i(\omega t - kz + \phi(z,t))}\right) \\
&= i\omega\frac{1}{\varepsilon_0 c^2}\left(-i\omega\mathcal{P} + \cancel{\frac{\partial \mathcal{P}}{\partial t} - \frac{\partial \phi}{\partial t}}\mathcal{P}\right)e^{-i(\omega t - kz + \phi(z,t))} = \frac{\omega^2}{\varepsilon_0 c^2}\mathcal{P}e^{-i(\omega t - kz + \phi(z,t))}
\end{aligned}
$$

Overall, if we simplify the phase we arrive to

$$
\frac{\partial \mathcal{E}}{\partial z} + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} + i\left(k - \frac{\omega}{c} - \frac{\partial \phi}{\partial z} - \frac{1}{c}\frac{\partial \phi}{\partial t}\right)\mathcal{E} = i\frac{\omega^2}{2k\varepsilon_0 c^2}\mathcal{P} \approx i\frac{\omega}{2\varepsilon_0 c}\mathcal{P}
$$

Note that we do not use the linear dispersion on the "$k - \frac{\omega}{c}$" term to allow us to derive the corrections to this relation.

Semiclassical model : effects of complex susceptibility

Taking the real and imaginary part ($\mathcal{P}$ is complex in general) and using $\mathcal{P} = \varepsilon_0 \mathcal{E} \chi(\omega) = \varepsilon_0 \mathcal{E} \left(\chi'_{\mathrm{Re}} + i\chi''_{\mathrm{Im}}\right)$ gives

Physical meaning of the real and imaginary parts of susceptibility

$$\text{(absorption/gain)} : \frac{\partial \mathcal{E}}{\partial z} + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} = -\frac{k}{2\varepsilon_0}\mathrm{Im}\mathcal{P} = -\frac{k}{2}\cdot \chi''_{\mathrm{Im}} \cdot \mathcal{E}$$

$$\text{(dispersion)} : \frac{\partial \phi}{\partial z} + \frac{1}{c}\frac{\partial \phi}{\partial t} = k - \frac{\omega}{c} - \frac{\omega}{2\varepsilon_0 c}\frac{\mathrm{Re}\mathcal{P}}{\mathcal{E}} = k - \left(1 + \frac{\chi'_{\mathrm{Re}}}{2}\right)\frac{\omega}{c}$$

The term $g(\omega) = -\frac{k}{2} \cdot \chi''_{\mathrm{Im}}(\omega)$ is the gain coefficient. We also have the corrected dispersion relation when we assume a constant phase ϕ :

$$k = \left(1 + \frac{\chi'_{\mathrm{Re}}}{2}\right)\frac{\omega}{c}$$

Semiclassical model : complex refractive index

If we consider a complex refractive index given by $n^2(\omega) = 1 + \chi$, up to first order approximation we have:

$$n(\omega) = n' + in'' = \sqrt{1 + \chi(\omega)} \approx 1 + \frac{\chi'_{\mathrm{Re}}}{2} + i\frac{\chi'_{\mathrm{Im}}}{2}$$

Hence $n' \approx 1 + \frac{\chi''_{\mathrm{Re}}}{2}$ and $n'' \approx \frac{\chi''_{\mathrm{Im}}}{2}$. Coming back to the dispersion relation, we see that the real part of the refractive complex index corresponds to the classical refractive index which

$$n' = \left(1 + \frac{\chi'_{\mathrm{Re}}}{2}\right) = c\frac{k}{\omega} = \frac{c}{v_p}$$

We recover the classical connection between the refractive index and the ratio of the speed of light in vacuum c to the phase velocity $v_p = \frac{\omega}{k}$ in the medium.

Semiclassical model : complex susceptibility

Solving the Bloch equations gives us

Susceptibility

$$\chi'_{\text{Re}} = N\frac{|\mathbf{p}_{12}|^2}{\varepsilon_0\hbar}\frac{\Delta}{\Gamma_{12}^2+\Delta^2}(\rho_{11}-\rho_{22})$$

$$\chi''_{\text{Im}} = N\frac{|\mathbf{p}_{12}|^2}{\varepsilon_0\hbar}\frac{\Gamma_{12}}{\Gamma_{12}^2+\Delta^2}(\rho_{11}-\rho_{22})$$

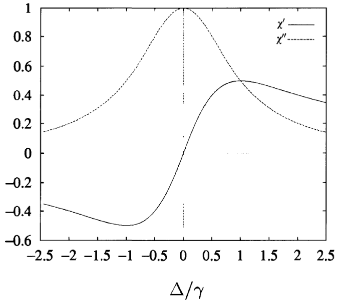


Figure: Real and imaginary part of the susceptibility

Generalization : Susceptibility of multilevel atoms

$$\chi = \frac{\mathcal{P}}{\varepsilon_0\mathcal{E}} = \frac{2}{\varepsilon_0\mathcal{E}}\left(\rho_{12}p_{21}+\rho_{13}p_{31}\right)e^{i\omega_0 t}$$

Slow and Fast light

Dispersion relation $\omega = \omega(k)$ and assuming $|\chi'_{\text{Re}}(\omega)| \gg |\chi''_{\text{Im}}(\omega)|$ such that the refractive index is $n(\omega) \approx n'(\omega) = \frac{c}{v_p} = \frac{ck}{\omega}$.

$$\text{Phase velocity : } v_p = \frac{\omega}{k}$$

$$\text{Group velocity : } v_g = \frac{d\omega}{dk} = \left(\frac{dk}{d\omega}\right)^{-1}$$

Eliminate k and rewrite group velocity in terms of refractive index

$$v_g = \frac{c}{n_g}, \text{ where } n_g = n(\omega) + \omega\frac{dn(\omega)}{d\omega} \text{ is the "Group Index"}$$

Now, we go back to the definition of polarizability

$$P(\omega) = \varepsilon_0 \chi(\omega) E \implies n(\omega) = \sqrt{1 + \chi'_{\text{Re}}(\omega)} \approx 1 + \frac{\chi'_{\text{Re}}(\omega)}{2}$$

By modifying $\chi(\omega)$, you can modify $n(\omega)$ and the group velocity of the light in the medium.

Slow and Fast light

We can engineer the $\chi(\omega)$ of the system such that the group velocity is-

- $v_g \ll c$ - "Slow light", Eg. EIT[5] gives[6] $v_g \approx$ 17 m/s
- $v_g > c$ or $v_g < 0$ - "Fast or Advanced light", Anomalous Dispersion by two Raman gain resonances [7]

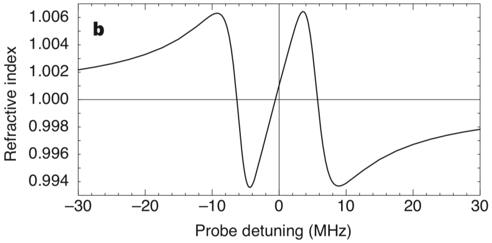


Figure: Refractive index profile. The steepness of the slope at resonance is inversely proportional to the group velocity of transmitted light

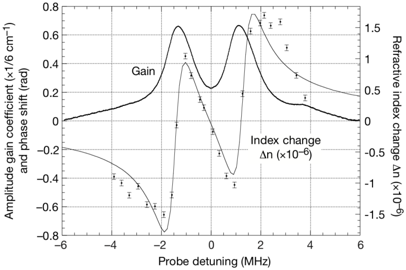


Figure: Measured refractive index and gain coefficient.

[5] Substitute numerical values in HW7.5 from Ch. 5 of Fast light, slow light and Left-Handed light, PW Milonni, 2005

[6] Hau, L. et al. Light speed reduction to 17 metres per second in an ultracold atomic gas. Nature 397, 594-598 (1999)

[7] Gain-assisted superluminal light propagation, L. J. Wang, A. Kuzmich & A. Dogariu, *Nature* **406** (2000)

Slow and Fast light

What about causality?[8]

- Phase velocity: $v_p > c$ allowed
- Group velocity: $v_g > c$ allowed. Not the same as velocity of information travel. Can be explained by Classical theory of wave propagation.
- Velocity of energy transfer: $v_E = |S(\omega)|/u(\omega)$, where S is the Poynting vector and u the Energy density. One can show that $v_E \leq c$. More interpretive than measurable.

[8] Fast light, slow light and Left-Handed light, PW Milonni, 2005

Quantum theory of atom-field interaction

The quantized electic field is

$$\hat{\mathbf{E}}(\mathbf{r},t) = \mathcal{E}_k \varepsilon_{\mathrm{ZPF}} \left(\hat{a}_k e^{-i\omega_0 t + i\mathbf{k}\cdot\mathbf{r}} + \hat{a}_k^\dagger e^{+i\omega_0 t - i\mathbf{k}\cdot\mathbf{r}} \right),$$

where $\varepsilon_{\mathrm{ZPF}} = \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V_k}}$ and $\mathcal{E}_k$ is the polarization

$$\hat{H}_{\mathrm{int}} = e\hat{\mathbf{r}} \cdot \hat{\mathbf{E}} = e\hat{\mathbf{r}} \cdot \mathcal{E}_k \varepsilon_{\mathrm{ZPF}} \left(\hat{a}_k e^{-i\omega_0 t + i\mathbf{k}\cdot\mathbf{r}} + \hat{a}_k^\dagger e^{+i\omega_0 t - i\mathbf{k}\cdot\mathbf{r}} \right)$$

$$\begin{aligned} e\hat{\mathbf{r}} &= e\mathbb{1}\hat{\mathbf{r}}\mathbb{1} = e(|1\rangle\langle 1| + |2\rangle\langle 2|)\hat{\mathbf{r}}(|1\rangle\langle 1| + |2\rangle\langle 2|) \\ &= e\underbrace{\langle 1|\hat{\mathbf{r}}|2\rangle}_{\mathbf{p}_{12}}\underbrace{|2\rangle\langle 1|}_{\hat{\sigma}^+} + e\underbrace{\langle 1|\hat{\mathbf{r}}|1\rangle}_{\text{zero}}|2\rangle\langle 1| + \underbrace{e\langle 2|\hat{\mathbf{r}}|1\rangle}_{\mathbf{p}_{21}=\mathbf{p}_{12}}\underbrace{|1\rangle\langle 2|}_{\hat{\sigma}^-} \end{aligned}$$

Atom-field interaction

$$\hat{H}_{\mathrm{int}} = (\mathbf{p}_{12}\hat{\sigma}^+ + \mathbf{p}_{21}\hat{\sigma}^-) \cdot \mathcal{E}_k \varepsilon_{\mathrm{ZPF}} \left(\hat{a}_k e^{-i\omega_0 t} + \hat{a}_k^\dagger e^{+i\omega_0 t} \right)$$

$$\hat{\sigma}^+ \hat{a}_k |1,n\rangle = |2,n-1\rangle \quad , \quad \hat{\sigma}^- \hat{a}_k |2,n\rangle = |1,n-1\rangle \quad , \quad \Delta E = \hbar(\omega_{12} + \omega_0)$$

Quantum theory of atom-field interaction

Rotating wave approximation

$$\hat{H}_{\text{int}} = \hbar\omega_{12}\frac{\hat{\sigma}_z}{2} + \hbar\sum_k g_k(\hat{\sigma}^+\hat{a}_k + \hat{\sigma}^-\hat{a}_k^\dagger)$$

$$g_k = \frac{\mathbf{p}_{12}\cdot\mathcal{E}_k\varepsilon_{\text{ZPF}}}{\hbar} = \mathbf{p}_{12}\cdot\mathcal{E}_k\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V_k}}\frac{1}{\hbar}$$

Rabi frequency becomes photon-number dependent in the quantum mechanical atom-field interaction.

$$\langle 1,n|\,\hbar g_k\hat{\sigma}^-\hat{a}_k^\dagger\,|2,n-1\rangle = \langle 1|\,\hat{\sigma}^-\,|2\rangle\,\langle n|\,\hat{a}_k^\dagger\,|n-1\rangle\,\hbar g_k = \hbar\sqrt{n}g_k$$

$$\langle 2,n+1|\,\hbar g_k\hat{\sigma}^+\hat{a}_k\,|1,n\rangle = \langle 2|\,\hat{\sigma}^+\,|1\rangle\,\langle n-1|\,\hat{a}_k\,|n\rangle\,\hbar g_k = \hbar\sqrt{n}g_k$$

However $\hat{\sigma}^+\hat{a}_k^\dagger$ and $\hat{\sigma}^-\hat{a}_k$ do not conserve excitation :

$$\hat{\sigma}^+\hat{a}_k^\dagger\,|1,n\rangle = \sqrt{n}\,|2,n+1\rangle \text{ thus } \Delta E = -\hbar(\omega+\omega_{12})$$
$$\hat{\sigma}^-\hat{a}_k\,|2,n\rangle = \sqrt{n}\,|1,n-1\rangle \text{ thus } \Delta E = +\hbar(\omega+\omega_{12})$$

Quantum theory of atom-field interaction : state evolution

Consider the manifold spanned by $\{|1\rangle|n+1\rangle, |2\rangle|n\rangle\}$ and ω_0 as laser frequency, $E_1 = \hbar\omega_1, E_2 = \hbar\omega_2$, where a generic state in this subspace can be written as:

$$|\Psi(t)\rangle = c_1(t)|1\rangle|n+1\rangle + c_2(t)|2\rangle|n\rangle$$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}c_1 & = -ig\sqrt{n+1}c_2 e^{+i(\omega_0-\omega_1)t} \\ \frac{\mathrm{d}}{\mathrm{d}t}c_2 & = -ig\sqrt{n+1}c_1 e^{-i(\omega_0-\omega_2)t} \end{cases}$$

As for the semi-classical case, we have Rabi oscillations, but now with a frequency :

$$\Omega_n^2 = \Delta^2 + 4g^2(n+1)$$

Now, we will perform a change of frame into the interaction picture, using Baker-Campbell-Hausdorff formula

$$e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} \approx \hat{B} + \alpha[\hat{A},\hat{B}] + \alpha^2[\hat{A},[\hat{A},\hat{B}]]/2!$$

Quantum theory of atom-field interaction : interaction picture

The time evolution of $|\Psi\rangle$ under the bare Hamiltonian $\hat{H}_0$

$$i\hbar\partial_t |\Psi\rangle = \hat{H}_0 |\Psi\rangle \ , \ \big|\Psi(t)\big\rangle = \big|\Psi(0)\big\rangle e^{-i\hat{H}_0 t/\hbar}$$

Moving to **interaction picture**, we remove this time evolution from the state and put it on the operators instead

$$\langle\Psi|\, \hat{H}_{\text{int}} |\Psi\rangle = \langle\Psi_0|\, e^{+i\hat{H}_0 t/\hbar}\hat{H}_{\text{int}} e^{-i\hat{H}_0 t/\hbar} |\Psi_0\rangle$$

The operators now transform as the Hamiltonian $\langle\Psi|\, \hat{a} |\Psi\rangle = \langle\Psi_0|\, e^{+i\hat{H}_0 t/\hbar}\hat{a} e^{-i\hat{H}_0 t/\hbar} |\Psi_0\rangle$

$$\begin{cases} \hat{a} & \rightarrow e^{i\omega_0 \hat{a}^\dagger \hat{a} t}\, \hat{a}\, e^{-i\omega_0 \hat{a}^\dagger \hat{a} t} & = \hat{a} e^{-i\omega_0 t} \\ \hat{\sigma}^+ & \rightarrow e^{i\omega_{12}\hat{\sigma}_z t/2}\, \hat{\sigma}^+\, e^{-i\omega_{12}\hat{\sigma}_z t/2} & = \hat{\sigma}^+ e^{-i\omega_{12} t} \end{cases}$$

Quantum theory of atom-field interaction : dressed states

The Hamiltonian in the interaction picture is for $\Delta = \omega_0 - \omega_{12}$

Interaction Hamiltonian in rotating frame

$$\hat{H}_{\text{int}} = \hbar g \left(\hat{\sigma}^+ \hat{a} e^{-i\Delta t} + \hat{\sigma}^- \hat{a}^\dagger e^{+i\Delta t} \right)$$

Dressed states of atom-field interaction

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|n,2\rangle + |n+1,1\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}} (|n,2\rangle - |n+1,1\rangle)$$

Spontaneous emission: Wigner-Weisskopf Theory

Assume that

$$|\psi(t=0)\rangle = \sum_k |0_k, 2\rangle$$
$$|\psi(t)\rangle = \sum_k c_1 |0_k, 2\rangle + \underbrace{\sum_k c_{2,k} |1_k, 1\rangle}_{\text{all possible modes k of the field}}$$

Thus we obtain

$$\begin{cases} \dot{c}_1 & = -i \sum_k g_k e^{i(\omega_{12}-\omega_k)t} c_{2,k} \\ \dot{c}_{2,k} & = i g_k e^{-i(\omega_{12}-\omega_k)t} c_1 \end{cases}$$

Eliminating $c_{2,k} = -i g_k(r_0) \int_0^t \mathrm{d}t' e^{-i(\omega_{12}-\omega_k)t} c_1(t')$:

$$\dot{c}_1 = \sum_k |g_k(r)|^2 \int_0^t \mathrm{d}t' e^{i(\omega_{12}-\omega_k)(t-t')} c_1(t')$$

Spontaneous emission: Wigner-Weisskopf Theory

Next we introduce the **density of states** (note that $\omega = k \cdot c$ and the factor 2 due to polarizations):

$$\sum_k \Rightarrow 2\frac{V}{(2\pi)^3}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\sin\theta\int\mathrm{d}k\cdot k^2$$
$$= 2\frac{V}{8\pi^3 c^3}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\sin\theta\int\mathrm{d}\omega_k\cdot\omega_k^2$$

Note that $|g(\vec{r})|^2 = \frac{\hbar\omega_k}{2\varepsilon_0 V}|\mathbf{p}_{12}|^2\cos^2\theta$ where θ is the angle between $\mathcal{E}_k$ and $\mathbf{p}_{12}$. Hence one obtains:

$$\dot{c}_1(t) = -\frac{4|\mathbf{p}_{12}|^2}{(2\pi)^2 6\hbar\varepsilon_0 c^3}\int_0^{\infty}\mathrm{d}\omega_k\omega_k^3\int_0^t\mathrm{d}t' e^{i(\omega_{12}-\omega_k)(t-t')}c_1(t')$$

Spontaneous emission: Wigner-Weisskopf Theory

Note that $\int_{-\infty}^{+\infty} \mathrm{d}\omega_k e^{-i(\omega_{12}-\omega_k)(t-t')} = 2\pi\delta(t-t')$, and $\int_0^t \delta(t-t')\mathrm{d}t' = \frac{1}{2}$, we have:

$$\dot{c}_1(t) = -\left[\frac{1}{4\pi\varepsilon_0}\frac{4\omega^3|\mathbf{p}_{12}|^2}{3\hbar c^3}\right]\frac{1}{2}c_1(t)$$

Spontaneous emission rate

$$\Gamma_{12} = \frac{1}{4\pi\varepsilon_0}\frac{4\omega^3|\mathbf{p}_{12}|^2}{3\hbar c^3}$$

Paper for this week

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Fast reset and suppressing spontaneous emission of a superconducting qubit

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Sythesizing arbitary quantum states in superconducting resonator

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LETTERS

Synthesizing arbitrary quantum states in a superconducting resonator

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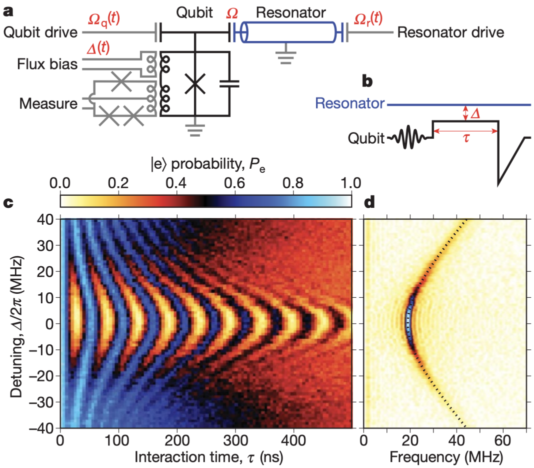
Sythesizing arbitary quantum states in superconducting resonator



Sythesizing arbitary quantum states in superconducting resonator

Table 1 | Sequence to generate the resonator state $|\psi\rangle = |1\rangle + i|3\rangle$

Sequence of states, operations	Operational parameter	System state, parameter value
$\|\psi\rangle$		$\|g\rangle(0.707\|1\rangle+0.707i\|3\rangle)$
S_3	$\tau_3\Omega$	1.81
Q_3	q_3	3.14
$\|\psi_2\rangle$		$\|g\rangle(-0.557i\|0\rangle+0.707\|2\rangle)+0.436\|e\rangle\|1\rangle$
Z_2	$t_2\Delta$	4.71
S_2	$\tau_2\Omega$	1.44
Q_2	q_2	$-2.09 - 2.34i$
$\|\psi_1\rangle$		$(0.553-0.62i)\|g\rangle\|1\rangle-(0.371+0.416i)\|e\rangle\|0\rangle$
Z_1	$t_1\Delta$	3.26
S_1	$\tau_1\Omega$	1.96
Q_1	q_1	$-2.71-1.59i$
$\|\psi_0\rangle$		$(0.197-0.98i)\|g\rangle\|0\rangle$

This resonator state is used for the measurements described in Fig. 2. The sequence is computed top to bottom, but applied bottom to top. The area and phase for the nth qubit drive Q_n is $q_n = \int \Omega_q(t)e^{i\Delta_{\rm off}t}dt$ ($t = 0$ being the time when the qubit is tuned into resonance directly after the step Q_n), the time on-resonance for the qubit–resonator swap operation S_n is τ_n, and the time off-resonance (mod $2\pi/\Delta$) for the phase rotation Z_n is t_n. We note that the initial state $|\psi_0\rangle$ differs by an overall phase factor from the ground state $|g\rangle\,|0\rangle$, but this is not detectable. State descriptions are shown bold; operations are not in bold.

Sythesizing arbitary quantum states in superconducting resonator

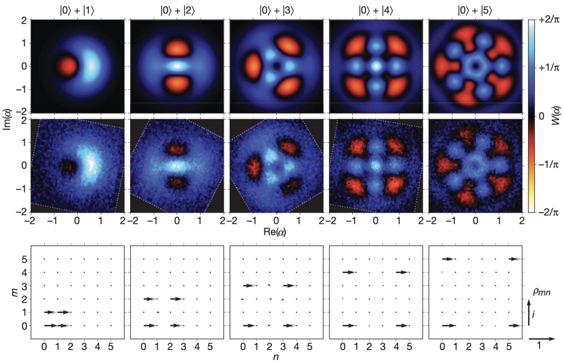


Figure 3 | Wigner tomography of superpositions of resonator Fock states $|0\rangle + |N\rangle$. The top row displays the theoretical form of the Wigner function $W(\alpha)$ as a function of the complex resonator amplitude α in photon number units, for states $N = 1$ to 5. The measured Wigner functions are shown in the middle row, with the colour scale bar on the far right. Negative quasi-probabilities are clearly measured. The experimental Wigner functions have been rotated to match theory, compensating for a phase delay between the qubit and resonator microwave lines; the measured area is bounded by a dotted white line. The bottom row displays the calculated (grey) and measured (black) values for the resonator density matrix ρ, projected onto the number states $\rho_{mn} = \langle m|\rho|n\rangle$. The magnitude and phase of ρ_{mn} is represented by the length and direction of an arrow in the complex plane (for scale, see key on right). The fidelities $F = \sqrt{\langle\psi|\rho|\psi\rangle}$ between the desired states $|\psi\rangle$ and the measured density matrices ρ are, from left to right, $F = 0.92, 0.89, 0.88, 0.94$ and 0.91. Each of the 51 by 51 pixels (61 by 61 for $N = 5$) in the Wigner function represents a local measurement. The value of $W(\alpha)$ is calculated at each pixel from 50 (41 for $N = 4$ and 5) interaction times τ, each repeated 900 times to give $P_e(\tau)$. This direct mapping of the Wigner function takes $\sim 10^8$ measurements or ~ 5 h.

Questions

- What is Purcell effect in atom radiative decay?
- How is Purcell factor related to the $\mathrm{Re}[Y(w)]$ in the circuit model? How is the admittance Y defined?
- Without Purcell filter, explain how different cavity kappa results in a trade off between qubit readout fidelity and life time.
- How is the admittance $Y(w)$ modified to suppress decay at low frequencies? What's the expression of the admittance?
- Why are two symmetrical $\lambda/4$ used instead of one? What's the difference to $\lambda/2$ resonator in terms of resonance frequencies and field distributions?
- What's the difference between static & pulsed T_1, why do they need two? Is the optimal point observable in the experiment? What are the other loss channels?