Quantum Electrodynamics and Quantum Optics: Lecture 7

Fall 2025

Description of atom-field interaction

Semi-classical

$\mathbf{E}(t,\mathbf{r}), \psi(t,\mathbf{r})$ **E** is a vector

Quantum

$\hat{\mathbf{E}}(t,\mathbf{r}), \psi(t,\mathbf{r})$ $\hat{\mathbf{E}}$ is an Operator

Quantum model predicts all effects e.g., Wigner-Weisskopf model of spontaneous emission, Lamb shift.

Semi-Classical model

Schrödinger equation: $\hat{H}\psi_j = \left(-\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{r})\right)\psi_j = i\hbar\frac{\partial}{\partial t}\psi_j$

The action of EM field is given by the Lorentz force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, which modifies the Hamiltonian as

$$\hat{H} = \frac{1}{2m}(\hat{\mathbf{p}} - q\mathbf{A})^2 + qU(\mathbf{r},t) + V(\mathbf{r})$$

where **A** is the vector potential and U is the Coulomb potential

Gauge transform

Hamiltonian can be rewritten from local (phase) gauge invariance:

$$\psi'(\mathbf{r},t) = e^{i\chi(\mathbf{r},t)}\psi(\mathbf{r},t)$$

$$\mathbf{A}' \leftarrow \mathbf{A} - \frac{\hbar}{q}\nabla\chi, \quad U' \leftarrow U + \frac{\hbar}{q}\partial_t\chi$$

In the Coulomb gauge, the free field U will vanish, leaving only the static Coulomb potential V of the atom.

Dipole approximation

$$\left(-\frac{\hbar^2}{2m}\left(\nabla+\frac{iq\mathbf{A}(\mathbf{r},t)}{\hbar}\right)^2+V(\mathbf{r})\right)\;\psi=i\hbar\partial_t\psi$$

Note that $|\psi(\mathbf{r})|^2$ is localized around a_0 (Bohr radius) and $a_0 \ll \lambda$. Therefore we have:

$$\mathbf{A}(\mathbf{r}+\mathbf{r}_0,t)=\mathbf{A}(0,t)e^{i\mathbf{k}\cdot(\mathbf{r}+\mathbf{r}_0)}\approx\mathbf{A}(0,t)e^{i\mathbf{k}\cdot\mathbf{r}_0}(1+\underbrace{\mathbf{k}\cdot\mathbf{r}+...}_{neglected})\approx\mathbf{A}(\mathbf{r}_0,t)$$

Applying the approximation:

$$\left(-\frac{\hbar^2}{2m}\left(\nabla+\frac{iq\mathbf{A}(\mathbf{r}_0,t)}{\hbar}\right)^2+V(\mathbf{r})\right)\psi(\mathbf{r})=i\hbar\partial_t\psi(\mathbf{r})$$

Light-matter r.E interaction Hamiltonian

Local gauge transformation and $\mathbf{r}\cdot\mathbf{E}$ Hamiltonian

Consider a gauge transformation of a wavefunction: $\psi(\mathbf{r},t) = -\phi(\mathbf{r},t)e^{i\chi(\mathbf{r},t)}$, which does not affect the probabilities, i.e. $|\psi|^2 = |\phi|^2$.

We choose the Gṗpert-Mayer gauge $\chi(\mathbf{r},t) = q/\hbar\mathbf{A}(r_0,t)\cdot\mathbf{r}$, such that inserting the wavefunction $\psi(\mathbf{r},t)$ in Schrödinger equation (with the dipole previoius approximation) yields:

$$i\hbar\left(\frac{iq}{\hbar}\right)\underbrace{\partial_t\mathbf{A}(r_0,t)}_{\mathbf{E}(r_0,t)}\cdot\mathbf{r}\phi(\mathbf{r},t) + i\hbar\partial_t\phi(\mathbf{r},t) = \underbrace{\left(\frac{\hat{p}^2}{2m} + V(\mathbf{r})\right)}_{\hat{H}_0}\phi(\mathbf{r},t),$$

which leads to a new form of total Hamiltonian:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}},$$

where $H_{\text{int}} = q\hat{\mathbf{r}}\cdot\mathbf{E}(r_0,t)$ is the rE-interaction term

Light-matter $\mathbf{p}.\mathbf{A}$ interaction Hamiltonian

$\mathbf{p} \cdot \mathbf{A}$ Hamiltonian

Upon transformation $p \to p - q\mathbf{A}(r_0, t)$, one can obtain:

$$\hat{H} = \frac{1}{2m}\left[i\hbar\nabla - q\mathbf{A}(r_0, t)\right]^2 + V(\mathbf{r}) = \hat{H}_0 + \hat{H}_1 + \hat{H}_2,$$

where

$\hat{H}_0 = -\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{r})$ is a free-electron Hamiltonian,

$\hat{H}_1 = \frac{i\hbar\nabla}{m} \cdot q\mathbf{A}(r_0, t) \propto \hat{\mathbf{p}} \cdot \mathbf{A}$ is a "pA"-interaction term, and

$\hat{H}_2 = [q\mathbf{A}(r_0, t)]^2/2m$ is the kinetic energy of electron induced by a field (neglected)

We obtain two terms in the Hamiltonian:

$$\hat{H}_{\text{int}}^{rE} = q\mathbf{r} \cdot \mathbf{E}(r_0, t) \text{ and } \hat{H}_{\text{int}}^{pA} = \mathbf{p} \cdot \mathbf{A}(r_0, t)/2m$$

Dipole approximation of two-level atomic system

Consider a two-level atom:

$$\hat{H}\,|1\rangle = \hbar\omega_1\,|1\rangle\,,\quad \hat{H}\,|2\rangle = \hbar\omega_2\,|2\rangle$$

We can calculate matrix elements on light-matter interaction Hamiltonian:

$$\begin{aligned}\langle 1|\,\hat{H}_{\text{int}}\,|1\rangle &= \iint \langle 1|\,\mathbf{r}\rangle\,\langle \mathbf{r}|\,\hat{H}_{\text{int}}\,|\mathbf{r}'\rangle\,\langle \mathbf{r}'\,|1\rangle\,\mathrm{d}^3r\,\mathrm{d}^3r' \\ &= \int \underbrace{\phi_1^*(\mathbf{r})\phi_1(\mathbf{r})}_{\text{even}}\,q\underbrace{\mathbf{r}}_{\text{odd}}\cdot\mathbf{E}(r_0,t)\mathrm{d}^3r \approx 0\end{aligned}$$

$$\langle 2|\,\hat{H}_{\text{int}}\,|2\rangle = 0$$

$$\langle 1|\,\hat{H}_{\text{int}}\,|2\rangle = \underbrace{\int \phi_1^*(\mathbf{r})\;\phi_2(\mathbf{r})\;q\;\mathbf{r}\;\mathrm{d}^3r}_{\mathbf{P}_{12}-\text{ matrix element of dipole moment}} \cdot\mathbf{E}(r_0,t) = \mathbf{P}_{12}\cdot\mathbf{E}(r_0,t)$$

$\phi_1(\mathbf{r})$ and $\phi_2(\mathbf{r})$ are wave functions with different spatial parity

Ladder operators between fermionic levels

Two-level systems obey the fermionic anti-commutation relations:

$$\{\hat{a}, \hat{a}^\dagger\} = 1$$
$$\{\hat{a}, \hat{a}\} = \{\hat{a}^\dagger, \hat{a}^\dagger\} = 0$$

The ladder operator for the transitions between two fermionic levels $\hat{\sigma}^+ = \hat{a}_2^\dagger \hat{a}_1$ and $\hat{\sigma}^- = \hat{a}_1^\dagger \hat{a}_2$ will thus obey the same anti-commutation relations

Two-level system:

$$\hat{\sigma}^+ |1,0\rangle = |0,1\rangle = |e\rangle \equiv |2\rangle$$
$$\hat{\sigma}^- |0,1\rangle = |1,0\rangle = |g\rangle \equiv |1\rangle$$

Ladder operators between fermionic levels

Pseudo-spin operators

$\hat{a}_2^\dagger \hat{a}_1 = \hat{\sigma}^+$ annihilates electron 1, creates electron 2 $\equiv$ **excitation**

$\hat{a}_1^\dagger \hat{a}_2 = \hat{\sigma}^-$ annihilates electron 2, creates electron 1 $\equiv$ **de-excitation**

$$\frac{1}{2}\left(\hat{a}_2^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_1\right) = \hat{\sigma}^z = \frac{1}{2}[\hat{\sigma}^+, \hat{\sigma}^-] \equiv \textbf{population inversion}$$

Pseudo-spin operators in $|1\rangle$,$|2\rangle$-basis

$$\hat{\sigma}^+ = |2\rangle\langle 1|$$

$$\hat{\sigma}^- = |1\rangle\langle 2|$$

$$\hat{\sigma}^z = \frac{1}{2}\left(|2\rangle\langle 2| - |1\rangle\langle 1|\right)$$

Density matrix for two-level systems

Density matrix formalism

$$\hat{\rho} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \quad \text{is a density matrix of two-level system}$$

$$\rho_{22} - \rho_{11} = 2\langle \hat{\sigma}^z \rangle \text{ is the population inversion}$$

$$\rho_{12} = \langle \hat{\sigma}^+ \rangle \text{ and } \rho_{21} = \langle \hat{\sigma}^- \rangle \text{ are } \textbf{coherences}$$

Light-matter interaction: Semiclassical dynamics

Schrödinger equation for two-level atom

Consider a two-level system under a field $\mathbf{E}(t) = \mathcal{E}\cos(\omega_L t)$, with $\mathcal{E}$ the constant vector field intensity:

$$\hat{H}_0 = \hbar\omega_1 |1\rangle\langle 1| + \hbar\omega_2 |2\rangle\langle 2| = \hbar(\omega_2 - \omega_1)\hat{\sigma}_z + \underbrace{\hbar\frac{\omega_1+\omega_2}{2}\mathbb{1}}_{\text{does not matter for the dynamics}}$$

$$\equiv \hbar\omega_{21}\hat{\sigma}_z, \qquad \text{where } \omega_{21} = \omega_2 - \omega_1$$

$$\hat{H}_{\text{int}} = q\hat{\mathbf{r}}\cdot\mathbf{E} = \underbrace{\langle 1| q\hat{\mathbf{r}} |2\rangle}_{\mathbf{p}_{12}} |1\rangle\langle 2| \cdot \mathbf{E} + \underbrace{\langle 2| q\hat{\mathbf{r}} |1\rangle}_{\mathbf{p}_{12}^*} |2\rangle\langle 1| \cdot \mathbf{E}$$

$$= \mathbf{p}_{12}\cdot\mathbf{E}\,\hat{\sigma}^- + \mathbf{p}_{12}^*\cdot\mathbf{E}\,\hat{\sigma}^+, \qquad \text{where } \mathbf{p}_{12} = q\int \phi_1^*(\mathbf{r})\phi_2(\mathbf{r})\mathbf{r}\,\mathrm{d}^3r$$

Note that the diagonal elements of $\hat{H}_{\text{int}}$ vanish due to parity since $\langle i| q\hat{\mathbf{r}} |i\rangle = 0$.

Light-matter interaction: Semiclassical dynamics

Schrödinger equation for two-level atom

The state of the system (Schrödinger picture) $\left|\psi(t)\right\rangle = C_1(t)\left|1\right\rangle + C_2(t)\left|2\right\rangle$ evolves in time as follows:

$$i\hbar\partial_t\left|\psi(t)\right\rangle = \left(\hat{H}_0 + \hat{H}_{\text{int}}\right)\left|\psi(t)\right\rangle,$$

Projecting both sides on $\langle 1|$ or $\langle 2|$, defining the **Rabi frequency** $\Omega_R = \mathbf{p}_{12}\cdot\mathcal{E}/\hbar$ and $\Delta = \omega_{21} - \omega_L$ the **detuning** of laser field from atomic resonance, we obtain

$$\begin{cases} \frac{\mathrm{d}C_1}{\mathrm{d}t} &= -i\omega_1 C_1 + i\Omega_R\cos(\omega_L t)C_2 \\ \\ \frac{\mathrm{d}C_2}{\mathrm{d}t} &= -i\omega_2 C_2 + i\Omega_R^*\cos(\omega_L t)C_1, \end{cases}$$

Light-matter interaction: Semiclassical dynamics

Schrödinger equation for two-level atom: rotating frame 1

We eliminate the first terms by going to the rotating frame with the unitary transformation $\hat{U} = \exp\left(i\hat{H}_0 t/\hbar\right)$ of the state $\left|\psi\right\rangle$. This transformation affects the amplitudes C_i as

$$C_i \to C_i' = C_i e^{i\omega_i t} \text{ and equivalently } C_i = C_i' e^{-i\omega_i t}$$

since $\left|\psi\right\rangle \to \left|\psi'\right\rangle = \hat{U}\left|\psi\right\rangle$ and the Schrödinger equations can be reformulated in this new frame as:

$$\begin{cases} \frac{\mathrm{d}C_1'}{\mathrm{d}t} &= \frac{i\Omega_R}{2}\left[e^{-i(\omega_{21}-\omega_L)t} + e^{-i(\omega_{21}+\omega_L)t}\right] C_2' \\ \frac{\mathrm{d}C_2'}{\mathrm{d}t} &= \frac{i\Omega_R^*}{2}\left[e^{i(\omega_{21}-\omega_L)t} + e^{i(\omega_{21}+\omega_L)t}\right] C_1', \end{cases}$$

Light-matter interaction: Semiclassical dynamics

Rotating wave approximation (RWA)

Now we define $\Delta = \omega_{21} - \omega_L$ the **detuning** of laser field from atomic resonance and consider the case of $|\Delta| \ll \omega_{21} + \omega_L$. In this case, the terms rotating with $\omega_{21} + \omega_L$ average out and can be neglected. This is the **rotating wave approximation**

$$\begin{cases} \frac{\mathrm{d}C_1'}{\mathrm{d}t} & \approx \frac{i\Omega_R}{2} e^{+i\Delta t} C_2' \\ \frac{\mathrm{d}C_2'}{\mathrm{d}t} & \approx \frac{i\Omega_R^*}{2} e^{-i\Delta t} C_1', \end{cases}$$

When the laser field **is close to atomic resonance**:

- $\Delta \approx 0$, which corresponds to $e^{\pm i\Delta t}$ is an almost **stationary** term in equation for population dynamics
- $\omega_{21} + \omega_L \gg |\Delta|$, which is a **fast-oscillating** term that does not affect averaged dynamics

Light-matter interaction: Semiclassical dynamics

Schrödinger equation for two-level atom: rotating frame 2

We use another unitary transformation to go to the stationary frame using

$$\begin{cases} C_1' & \to \tilde{C}_1 = C_1' e^{-i\Delta/2t} \text{ and equivalently } C_1' = \tilde{C}_1 e^{i\Delta/2t} \\ C_2' & \to \tilde{C}_2 = C_2' e^{i\Delta/2t} \text{ and equivalently } C_2' = \tilde{C}_2 e^{-i\Delta/2t} \end{cases}$$

In this second new frame, we can write the equation in a matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} -\Delta & \Omega_R \\ \Omega_R & \Delta \end{pmatrix}\begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix}$$

Light-matter interaction: Semiclassical dynamics

General solution of Rabi problem in terms of population inversion

$$|C_1|^2 - |C_2|^2 = \left(\frac{\Delta^2 - \Omega_R^2}{\Omega^2}\right)\sin^2\left(\frac{\Omega t}{2}\right) + \cos^2\left(\frac{\Omega t}{2}\right),$$

where $\Omega = \sqrt{\Omega_R^2 + \Delta^2}$ is the detuning dependent Rabi frequency

Light-matter interaction: Semiclassical dynamics

Rabi oscillations

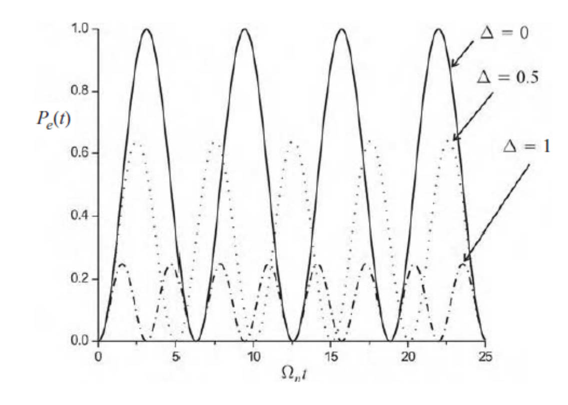

Figure: [1]Excited state population $P_e = |C_2(t)|^2$ dynamics for various detunings Δ

[1]Gerry, Christopher and Knight, Peter. *Introductory quantum optics*. Cambridge university press, 2005

Dipole moment

Considering atomic polarization $\hat{\mathbf{p}} = q\hat{\mathbf{r}}$:

$$\langle \hat{\mathbf{p}} \rangle = e \left\langle \Psi(t) \right| \hat{\mathbf{r}} \left| \Psi(t) \right\rangle = eC_2^* C_1 \langle 1 | \hat{\mathbf{r}} | 2 \rangle + eC_1^* C_2 \langle 2 | \hat{\mathbf{r}} | 1 \rangle = 2\mathrm{Re}\left[\mathbf{p}_{12} C_2^* C_1\right]$$

Recall that in original frame (2 rotations) we have: $(\Delta = 0, \Omega = \Omega_R)$

$$\begin{cases} C_1(t) = C_1(0)\cos\left(\frac{\Omega t}{2}\right)e^{-i\omega_1 t} + iC_2(0)\sin\left(\frac{\Omega t}{2}\right)e^{-i\omega_2 t} \\ C_2(t) = -iC_1(0)\sin\left(\frac{\Omega t}{2}\right)e^{-i\omega_1 t} + C_2(0)\cos\left(\frac{\Omega t}{2}\right)e^{-i\omega_2 t} \end{cases}$$

Dipole moment for $\Delta = 0$, assuming $C_1(0) = 1$ and $C_2(0) = 0$

$$\langle \hat{\mathbf{p}} \rangle = \mathrm{Re}\left[i\mathbf{p}_{12}\sin(\Omega t)\right]$$

Paper for this week

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LETTERS

Synthesizing arbitrary quantum states in a superconducting resonator

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Questions

- Explain the excitation swapping between qubit and the resonator shown in Fig.1c,d
- Describe the set of operations (Q,S,Z) used on qubit and resonator based on Eq(1)
- How is the sequence designed to synthesize an arbitrary state, how are complex state coefficients constructed
- How to do Wigner tomography on the synthesized state, how is the parity measurement performed
- What's the difference between $|0\rangle + |3\rangle$ and $|0\rangle + e^{i\cdot\phi}|3\rangle$ in terms of Wigner function, why does the relative phase of Fock states correspond to a global rotation of the Wigner function
- How are phases of Fock state superposition (density matrix) measured