Quantum Electrodynamics and Quantum Optics: Lecture 3

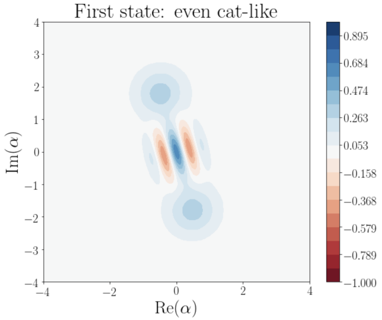
Fall 2025

QuTiP[1]

QuTiP
Quantum Toolbox in Python

```
In [8]: W_even=np.around(W_even, decimals=2)
        plt.figure(figsize=(10, 8))
        plt.contourf(xvec,xvec, W_even, cmap='RdBu', levels=np.linspace(-1,
         1, 20))
        plt.colorbar()
        plt.xlabel(r'Re$(\alpha)$', fontsize=label_size), width="300"
        plt.ylabel(r'Im$(\alpha)$', fontsize=label_size)
        plt.title("First state: even cat-like", fontsize=title_font)
        plt.show()
```

QuTiP is an open-source software for simulating the dynamics of open quantum system. QuTiP aims to provide user-friendly and efficient numerical simulations of a wide variety of Hamiltonians, including those with arbitrary time-dependence, commonly found in a wide range of physics applications such as quantum optics, trapped ions, superconducting circuits, and quantum nanomechanical resonators.



[1] Refer to http://qutip.org

QuTiP[2]

In a finite Hilbert space of Fock states one can define operators and immediatly obtain their matrix form, e.g. annihilation operator $\hat{a}$

```
a = destroy(5)
```

creation operator $\hat{a}^\dagger$

```
a.dag()
```

Quantum object: dims = [[5], [5]], shape = (5, 5), type = oper, isherm = False

$$\begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.414 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.732 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \end{pmatrix}$$

and compute the commutators $[\hat{a}, \hat{a}^\dagger]$

```
commutator(a, a.dag())
```

Quantum object: dims = [[5], [5]], shape = (5, 5), type = oper, isherm = True

$$\begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.000 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -4.0 \end{pmatrix}$$

A number operator $\hat{a}^\dagger \hat{a} = |n\rangle \langle n|$ in Hilbert space $|0\rangle, \ldots, |99\rangle$

```
a = destroy(100)
n = a.dag()*a
n
```

Quantum object: dims = [[100], [100]], shape = (100, 100), type = oper, isherm = True

$$\begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 2.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 3.000 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 4.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 95.000 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 96.000 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 97.000 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 98.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 99.0 \end{pmatrix}$$

and its expectation value in a coherent state

```
alpha = coherent(100, 2 + 3 * 1j)
print(expect(n, alpha))
```

```
12.999999999999998
```

[2] Refer to http://qutip.org

Squeezed States Review[3]

Squeezing operator

$$\hat{S}(\xi) = e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})}$$

We can also re-express $\xi = re^{i\theta}$ in terms of $\mu = \cosh r$ and $\nu = e^{i\theta} \sinh r$. Without giving full operator "disentangling" calculation, this is equal to the following normally ordered expression

$$\hat{S}(\xi) = e^{-\frac{\nu}{2\mu}\hat{a}^{\dagger 2}} \left(\frac{1}{\mu}\right)^{\hat{n}+\frac{1}{2}} e^{\frac{\nu^*}{2\mu}\hat{a}^2}.$$

Hence, the vacuum squeezed state can be expressed as

$$|\xi, 0\rangle = \frac{1}{\sqrt{\mu}} e^{-\frac{\nu}{2\mu}\hat{a}^{\dagger 2}} |0\rangle$$

which is also called two photon coherent state.

[3] Quantum Optics W. Vogel Chapter 3

Squeezed States Review

Bogoliubov operator

Consider the operator $\hat{\boldsymbol{\beta}} = \boldsymbol{\mu}\hat{\boldsymbol{a}} + \boldsymbol{\nu}\hat{\boldsymbol{a}}^\dagger$ where $|\mu|^2 - |\nu|^2 = 1$ or equivalently $\mu = \cosh r$ and $\nu = e^{i\theta}\sinh r$. Then $\hat{\beta}$ obeys the commutation relation $[\hat{\beta}, \hat{\beta}^\dagger] = 1$. This operator arises from the Bogoliubov transformation used in many-body theory and quantum optics.

Squeezed state as eigenstate of Bogoliubov operator

It can be shown that the squeezed coherent state $|\xi, \alpha\rangle$ is the eigenstate of the Bogoliubov operator $\hat{\beta}$, i.e. $\hat{\beta}\,|\xi, \alpha\rangle = \beta\,|\xi, \alpha\rangle$. This can be shown by using the transformation property of the annihilation operator under the squeezing operator.

The squeezed coherent state $|\xi, \alpha\rangle$ can then be expressed in two equivalent ways:
$|\xi, \alpha\rangle = \hat{S}(\xi)\hat{D}(\beta)\,|0\rangle = \hat{D}(\alpha)\hat{S}(\xi)\,|0\rangle$, where $\beta = \mu^*\alpha + \nu\alpha^*$.

Squeezed States Review

*A NEW METHOD IN THE THEORY OF SUPERCONDUCTIVITY. II**

V. V. TOLMACHEV and S. V. TIABLIKOV

Mathematics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor October 17, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 66-72 (January, 1958)

The equivalence of the Bardeen Hamiltonian and the Fröhlich Hamiltonian is established in the adiabatic approximation. The energy of the ground state and of elementary excitations are calculated by means of a canonical transformation.

Squeezed States Review

IN Ref. 1, Bogoliubov has shown that the property of superconductivity is possessed by a model of an electron gas in which the mutual interaction of the electrons is neglected but their interaction with the phonon field is taken into account. These results were established with the help of the Fröhlich Hamiltonian for the description of the system:[2]

$$H = H_{\text{el}} + H_{\text{int}} + H_{\text{ph}}; \tag{1}$$

$$H_{\text{el}} = \sum_{(k,\sigma)} (E(k) - \lambda)\, a^+_{k,\sigma} a_{k,\sigma};\quad H_{\text{ph}} = \sum_{(q)} \hbar\omega(q)\, b_q^+ b_q; \tag{2}$$

$$H^{\text{Fr}}_{\text{int}} = \frac{g}{\sqrt{2V}} \sum_{(k,k',\sigma)} \hbar\omega(k-k')(a^+_{k',\sigma} a_{k,\sigma} b_{k-k'} + a^+_{k,\sigma} a_{k',\sigma} b^+_{k'-k}), \tag{3}$$

where $E(k)$ is the energy of the electron; $\hbar\omega(q)$ the energy of the phonon; k, q are the wave vectors, σ the spin variable ($\sigma = \pm\frac{1}{2}$); V the volume of the system; g the coupling constant; and λ the chemical potential. The creation and annihilation operators of electrons (a^+, a) and phonons

We carry out a canonical transformation[1] on the operators

$$a_{k,1/2} = u_k\alpha_{k,1} + v_k\alpha^+_{-k,0};\quad a_{k,-1/2} = u_k\alpha_{k,0} - v_k\alpha^+_{-k,1};$$
$$a^+_{k,1/2} = u_k\alpha^+_{k,1} + v_k\alpha_{-k,0};\quad a^+_{k,-1/2} = u_k\alpha^+_{k,0} - v_k\alpha_{-k,1}; \tag{11}$$

$$u_k^2 + v_k^2 = 1;\quad (u_{-k} = u_k;\ v_{-k} = v_k), \tag{12}$$

where α, α^+ are new Fermi operators; u_k, v_k are the coefficients of the transformation, which will be defined below.

Squeezed States Review

*A NEW METHOD IN THE THEORY OF SUPERCONDUCTIVITY. I**

N. N. BOGOLIUBOV

Mathematics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor October 10, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 58-65 (January, 1958)

The canonical transformation method previously developed by the author for the theory of superfluids is generalized in the present paper. By application of this method and the principle of compensation of "dangerous" diagrams, it is shown that a superconducting state is inherent in the Fröhlich model. Computation of the principal parameters of this state leads to formulas that confirm those of the theory of Bardeen, Cooper, and Shrieffer.

Squeezed States Review

Squeezing operator : position / momentum basis

Starting from the transformation of the quadrature under the squeezing operator :

$$\hat{S}^\dagger(\xi)(\hat{Y}_1 + i\hat{Y}_2)\hat{S}(\xi) = \hat{Y}_1 e^{-r} + i\hat{Y}_2 e^{r},$$

we can derive the transformation of the $\hat{x}$ and $\hat{p}$ in the **Heisenberg picture**. Indeed

$$\hat{x}_{\text{sq}} = \hat{S}^\dagger(\xi)\hat{x}\hat{S}(\xi) = e^{-r}\hat{x}$$

$$\hat{p}_{\text{sq}} = \hat{S}^\dagger(\xi)\hat{p}\hat{S}(\xi) = e^{r}\hat{p}.$$

Now in the **Schrödinger picture**, this leads to rescaled coordinates

$$\psi_{\text{sq}}(x) = \langle x|\hat{S}(\xi)|\psi\rangle = e^{r/2}\psi(e^{r}x)$$

$$\tilde{\psi}_{\text{sq}}(p) = \langle p|\hat{S}(\xi)|\psi\rangle = e^{-r/2}\tilde{\psi}(e^{-r}p),$$

where $\tilde{\psi}$ is the Fourier Transform of ψ.

Squeezed States Review

Squeezing operator : position / momentum basis

This last step is straightforward using the resolution of identity $\mathbb{1} = \int \mathrm{d}x \, |x\rangle \langle x|$,

$$\langle \hat{x} \rangle_{\mathrm{sq}} = \langle \psi_{\mathrm{sq}} | \hat{x} | \psi_{\mathrm{sq}} \rangle = \int \mathrm{d}x \, x \left| \psi_{\mathrm{sq}}(x) \right|^2 = \langle \psi | \hat{S}^\dagger(\xi) \hat{x} \hat{S}(\xi) | \psi \rangle = e^{-r} \langle \hat{x} \rangle$$
$$= e^{-r} \langle \psi | \hat{x} | \psi \rangle = \int \mathrm{d}y \, y e^{-r} \left| \psi(y) \right|^2 = \int \mathrm{d}x \, x e^{r} \left| \psi(e^r x) \right|^2 = \int \mathrm{d}x \, x \left| e^{r/2} \psi(e^r x) \right|^2,$$

where we used $y = e^r x$.

Squeezing operator : Fock states

Since even Fock states are...even $\psi_{2n}(x) = \psi_{2n}(-x)$ and odd Fock states are odd $\psi_{2n}(x) = -\psi_{2n}(-x)$, and the squeezing operator is actually a **scaling operator for the coordinates** (does not modify the parity), odd component of a squeezed vacuum state vanish

Photon Number Distribution of a Squeezed State

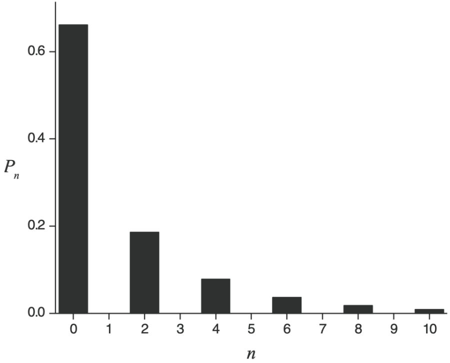
The photon number distribution

$$p_n = \langle n| \hat{S}(\xi) |0\rangle \tag{1}$$

In position representation

$$p_n = \int_{-\infty}^{\infty} \langle n|x\rangle \cdot \langle x| \hat{S}(\xi) |0\rangle \, dx = \tag{2}$$

$$= \int_{-\infty}^{\infty} \psi_n(x) \cdot e^{r/2} \psi_0(e^r x) dx \tag{3}$$



Since $\psi_n(x)$ is even (i.e. symmetric $\psi(x) = \psi(-x)$) for even n (including $n = 0$), p_n vanishes for odd n.

Quadrature Representation

Optical quadratures

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$$

$$\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger) \qquad [\hat{X}_1, \hat{X}_2] = \frac{i}{2}$$

or more generally

$$\hat{X}_\varphi = \hat{a}e^{i\varphi} + \hat{a}^\dagger e^{-i\varphi} \qquad [\hat{X}_\varphi, \hat{X}_{\varphi+\frac{\pi}{2}}] = 2i$$

The fluctuations of the quadrature[a] $\hat{X}_\varphi$ are $\left\langle \Delta\hat{X}_\varphi^2 \right\rangle \equiv \left\langle \hat{X}_\varphi^2 \right\rangle - \left\langle \hat{X}_\varphi \right\rangle^2$, and without proof:

$$\langle \beta, \xi | \, \Delta\hat{X}_\varphi^2 \, | \beta, \xi \rangle = |\mu e^{i\varphi} - \nu e^{-i\varphi}|^2 = |\mu - |\nu| e^{2i\varphi + \theta_\xi}|^2$$

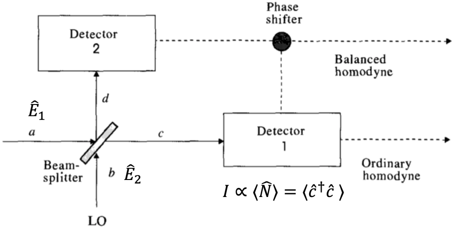
[a] note that different definitions exist e.g. $\frac{1}{2}$; $\frac{1}{\sqrt{2}}$; 1.

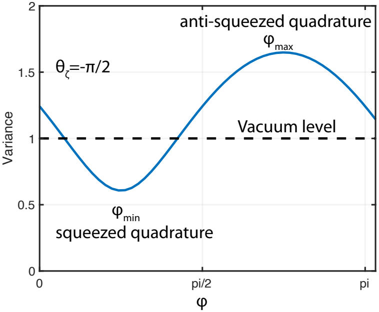
Quadrature Representation

$$\langle \beta, \xi | \, \Delta \hat{X}_{\varphi}^{2} \, | \beta, \xi \rangle = e^{2r}$$ enhanced fluctuations

$$\langle \beta, \xi | \, \Delta \hat{X}_{\varphi + \frac{\pi}{2}}^{2} \, | \beta, \xi \rangle = e^{-2r}$$ suppressed fluctuations

Can be measured using homodyne detection





Homodyne Detection[4]

$$\hat{E}_1(r,t) = i\left(\frac{\hbar\omega}{2V\epsilon_0}\right)^{1/2}\left(\hat{a}e^{-i(kr+\omega t)} - \hat{a}^\dagger e^{i(kr+\omega t)}\right)$$

The beamsplitter output operator obeys $\hat{c} = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{b}$. The photodetection current in the output mode $I_c \propto \langle\hat{N}\rangle = \langle\hat{c}^\dagger\hat{c}\rangle$ can then be calculated as:

$$I_c \propto \left\langle\hat{c}^\dagger\hat{c}\right\rangle \propto \eta\left\langle\hat{a}^\dagger\hat{a}\right\rangle + (1-\eta)\left\langle\hat{b}^\dagger\hat{b}\right\rangle + \sqrt{\eta(1-\eta)}(\langle\hat{a}\rangle\langle\hat{b}^\dagger\rangle - \langle\hat{a}^\dagger\rangle\langle b\rangle).$$

Notice that $\langle\hat{a}\hat{b}^\dagger\rangle = \langle\hat{a}\rangle\langle\hat{b}^\dagger\rangle$, assuming the states associated with $\hat{a}$ and $\hat{b}$ are uncorrelated. Let input mode $\hat{b}$ be in a relatively large coherent state $\left|\beta\right\rangle = \left||\beta|e^{i\varphi}\right\rangle$ compared to $\hat{E}_1$, one can measure arbitrary quadrature $\hat{X}_\varphi$:

$$\Rightarrow \left\langle\hat{c}^\dagger\hat{c}\right\rangle = (1-\eta)|\beta|^2 + |\beta|\sqrt{\eta(1-\eta)}\underbrace{\left\langle\hat{a}e^{-i\varphi} + \hat{a}^\dagger e^{i\varphi}\right\rangle}_{\propto\langle\hat{X}_\varphi\rangle}$$

[4] Scully, M.O., Zubairy, M.S. "Quantum optics" (1999). Chapter 4

Noise Properties of Squeezed State[5,6]

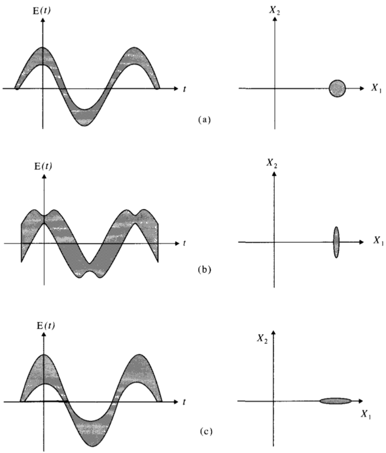


Fig. 2.7 Error contours and the corresponding graphs of electric field versus time for (a) a coherent state, (b) a squeezed state with reduced nosie in X_1, and (c) a squeezed state with reduced noise in X_2. (From C. Caves, *Phys. Rev. D* **23**, 1693 (1981).)

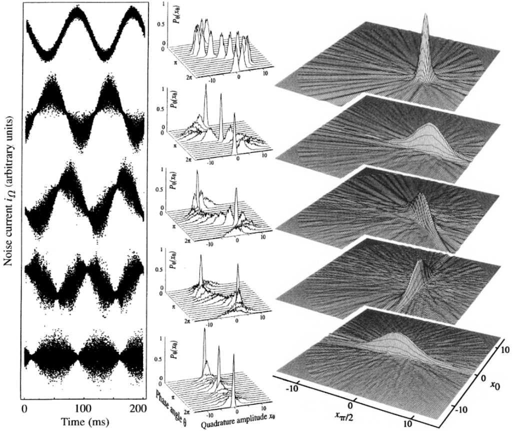


Figure 2 Noise traces in $i_\Omega(t)$ (left), quadrature distributions $P_\theta(x_\theta)$ (centre), and reconstructed Wigner functions (right) of generated quantum states. From the top: Coherent state, phase-squeezed state, state squeezed in the $\phi = 48°$-quadrature, amplitude-squeezed state, squeezed vacuum state. The noise traces as a function of time show the electric fields' oscillation in a 4π interval for the upper four states, whereas for the squeezed vacuum (belonging to a different set of measurements) a 3π interval is shown. The quadrature distributions (centre) can be interpreted as the time evolution of wave packets (position probability densities) during one oscillation period. For the reconstruction of the quantum states a π interval suffices.

[5] Quantum Optics - Marlan O. Scully, M. Suhail Zubairy - Chapter 2

[6] Breitenbach, G., Schiller, S. & Mlynek, J. Measurement of the quantum states of squeezed light. Nature 387, 471–475 (1997).

Review of Density Matrix[7]

Density matrix and probabilities

For a pure state $\left\langle \hat{M} \right\rangle = \langle \psi | \hat{M} | \psi \rangle, \quad \hat{\rho} \equiv \sum_i p(i) |\psi_i\rangle \langle \psi_i|$

For a mixed state $\left\langle \hat{M} \right\rangle = \sum_i p(i) \langle \psi | \hat{M} | \psi \rangle = \mathrm{Tr}(\hat{\rho}\hat{M})$.

$$\begin{aligned}\mathrm{Tr}(\hat{\rho}\hat{M}) &= \sum_n \sum_i p(i) \langle n|\psi_i\rangle \left\langle \psi_i|\hat{M}|n \right\rangle \\ &= \sum_n \sum_i p(i) \langle \psi_i | \hat{M} \underbrace{|n\rangle \langle n| \psi_i\rangle}_{\psi_i} = \sum p(i) \left\langle \psi_i|\hat{M}|\psi_i \right\rangle = \left\langle \hat{M} \right\rangle .\end{aligned}$$

[7] Stochastic Methods Gardiner Chapter 10

Review of Density Matrix

Density matrix and probabilities

Properties: $\mathrm{Tr}(\hat{A}\hat{B}\hat{C}) = \mathrm{Tr}(\hat{C}\hat{A}\hat{B})$ cyclic
(i) $\mathrm{Tr}\hat{\rho} = 1$ for $\sum_i p(i) \langle \psi_i | \psi_i \rangle = \sum p(i) = 1$
(ii) Pure state $\hat{\rho}^2 = \hat{\rho}$
Time evolution:

$$i\hbar \partial_t \hat{\rho} = [\hat{H}, \hat{\rho}] \quad \text{von Neumann Equation}$$

$$\hat{\rho}(t) = e^{-i\hat{H}t/\hbar} \hat{\rho}(0) e^{i\hat{H}t/\hbar}$$

We can express the density matrix in Fock states basis

$$\hat{\rho} = \sum_{n,m} \rho_{n,m} |n\rangle \langle m| = \sum \langle n| \hat{\rho} |m\rangle |n\rangle \langle m| .$$

Phase Space[8]

Alternatively, one can express it in coherent state basis by inserting $\mathbb{1} = \frac{1}{\pi}\int |\alpha\rangle\langle\alpha| d^2\alpha$, thus $\rho = \frac{1}{\pi^2}\iint d^2\alpha d^2\beta \rho(\alpha,\beta) |\alpha\rangle\langle\beta|$, where $\langle\alpha|\rho|\beta\rangle = \rho(\alpha,\beta)$ and for a normally ordered function $\hat{f}(\hat{a},\hat{a}^\dagger)$ we have

$$\left\langle \hat{f}(\hat{a},\hat{a}^\dagger)\right\rangle = \frac{1}{\pi^3}\iiint d^2\alpha d^2\beta d^2\gamma \langle\gamma \overbrace{|\alpha\rangle\langle\alpha|}^{\mathbb{1}} \rho |\beta\rangle\langle\beta|\hat{f}(\hat{a},\hat{a}^\dagger)|\gamma\rangle$$
$$\overset{\gamma\to\alpha}{=} \frac{1}{\pi^2}\iint d^2\alpha d^2\beta \rho(\alpha,\beta) f(\alpha,\beta^*)$$

R-representation

$R(\alpha^*,\beta) = \langle\alpha|\rho|\beta\rangle e^{\frac{1}{2}(|\alpha|^2+|\beta|^2)} = \sum_{n,m}\frac{\langle n|\rho|m\rangle}{\sqrt{n!m!}}\alpha^{*n}\beta^m$, thus:

$$\rho = \iint \frac{1}{\pi^2} d^2\alpha d^2\beta |\alpha\rangle\langle\beta| R(\alpha^*,\beta) e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}$$
$$\langle n|\rho|m\rangle = \frac{1}{\pi^2}\int R(\alpha^*,\beta)\sqrt{n!m!}^{-1}\alpha^n\beta^{*n} e^{-|\alpha|^2-|\beta|^2} d^2\alpha d^2\beta$$

[8] Glauber, Roy J. "Coherent and incoherent states of the radiation field". *Physical Review* 131.6 (1963): 2766.APA

Paper for next week's presentation

VOLUME 59, NUMBER 18 PHYSICAL REVIEW LETTERS 2 NOVEMBER 1987

Measurement of Subpicosecond Time Intervals between Two Photons by Interference

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Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

(Received 10 July 1987)

A fourth-order interference technique has been used to measure the time intervals between two photons, and by implication the length of the photon wave packet, produced in the process of parametric down-conversion. The width of the time-interval distribution, which is largely determined by an interference filter, is found to be about 100 fs, with an accuracy that could, in principle, be less than 1 fs.

PACS numbers: 42.50.Bs, 42.65.Re

Questions for the next week's paper presentation

1. How are the correlated-two-photon state defined? Can they be separated into a product of two single photon states?
2. Which element enables precise tuning of delay between pulses? In this setting, does the detector's timing resolution impose any limitation?
3. How to define coincidence in detection? How to describe the propagation of optical field in Heisenberg picture? How displacement of the BS appears as a delay in the field expression at the detectors?
4. What's the spatial coherence extent of the optical pulses and how they are determined from IF filters? What is the width of the dip feature in coincidence measurement, is it consistent with the spatial coherence length?
5. What are the physical mechanisms that result in lower interference visibility? (Photon Flux rate? Detector resolution? Detector noise (dark count)?)