Quantum Electrodynamics and Quantum Optics: Lecture 2

Fall 2025

Coherent States[1]

The positive frequency part of the electric field operator introduced by Glauber is given by

$$\hat{E}^{(+)}(\vec{r}) = i\sum_k \left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2} \hat{a}_k \vec{u}_k(\vec{r}) e^{-i\omega_k t}.$$

The eigenvalue function $\varepsilon(\vec{r})$ of the operator $\hat{E}^{(+)}(\vec{r})$, defined as a solution of the eigenvalue equation $\hat{E}^{(+)}(\vec{r})|\,\rangle = \varepsilon(\vec{r})|\,\rangle$, must also satisfy the Maxwell equations, just as the operator $\hat{E}^{(+)}(\vec{r})$ does. $\varepsilon(\vec{r})$ and $\hat{E}^{(+)}(\vec{r})$ therefore possess similar normal mode expansions. Introducing a set of c-number Fourier coefficients α_k we may write

$$\varepsilon(\vec{r}) = i\sum_k \left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2} \alpha_k \vec{u}_k(\vec{r}) e^{-i\omega_k t}$$

[1] Glauber, Roy J. "Coherent and incoherent states of the radiation field". *Physical Review* 131.6 (1963): 2766.APA

Coherent States[2]

Since the mode functions $\vec{u}_k(\vec{r})$ form an orthogonal set, it then follows that the eigenstate $|\ \rangle$ for the field $\hat{E}^{(+)}(\vec{r})$ obeys the infinite succession of relations

$$\hat{a}_k \,|\ \rangle = \alpha_k \,|\ \rangle\,.$$

The positive frequency part of the electric field operator is thus given, according to (2.10), by

$$\mathbf{E}^{(+)}(\mathbf{r}t) = i\sum_k \left(\tfrac{1}{2}\hbar\omega_k\right)^{1/2} a_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t}. \qquad (2.19)$$

The eigenvalue functions $\boldsymbol{\mathcal{E}}(\mathbf{r}t)$ defined by Eq. (2.2) must clearly satisfy the Maxwell equations, just as the operator $\mathbf{E}^{(+)}(\mathbf{r}t)$ does. They therefore possess an expansion in normal modes similar to Eq. (2.19). In other words we may introduce a set of c-number Fourier coefficients α_k which permit us to write the eigenvalue function as

$$\boldsymbol{\mathcal{E}}(\mathbf{r}t) = i\sum_k \left(\tfrac{1}{2}\hbar\omega_k\right)^{1/2} \alpha_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t}. \qquad (2.20)$$

Since the mode functions $\mathbf{u}_k(\mathbf{r})$ form an orthogonal set, it then follows that the eigenstate $|\ \rangle$ for the field obeys the infinite succession of relations

$$a_k|\ \rangle = \alpha_k|\ \rangle, \qquad (2.21)$$

for all modes k. To find the states which satisfy these relations we seek states, $|\alpha_k\rangle_k$, of the individual modes which individually obey the relations

$$a_k|\alpha_k\rangle_k = \alpha_k|\alpha_k\rangle_k. \qquad (2.22)$$

Coherent States

Coherent states

Coherent states $|\alpha\rangle$ are defined as the eigenstates of $\hat{a}$ with eigenvalue α with the following properties

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$$
$$\langle\alpha|\,\hat{a}\,|\alpha\rangle = \alpha \qquad \langle\alpha|\,\hat{a}^{\dagger}\,|\alpha\rangle = \alpha^{*}$$
$$\langle\alpha|\,\hat{a}^{\dagger}\hat{a}\,|\alpha\rangle = |\alpha|^{2}$$

In order to derive the expression of the coherent state in the Fock basis, we can use the definition in the following way

$$\langle n|\,\hat{a}\,|\alpha\rangle = \sqrt{n+1}\langle n+1\,|\alpha\rangle = \alpha\langle n\,|\alpha\rangle$$
$$\langle n|\alpha\rangle = \frac{\alpha^{n}}{(n!)^{1/2}}\,\langle 0|\alpha\rangle$$

Coherent States

We can then express the coherent state in the basis of Fock states as

$$|\alpha\rangle = \langle 0|\alpha\rangle \sum \frac{\alpha^n |n\rangle}{\sqrt{n!}}.$$

After normalization, we obtain the expression of a coherent state

Coherent states

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum \frac{\alpha^n |n\rangle}{\sqrt{n!}}$$

Note that the coherent states do not form an orthogonal basis, as

$$\left| \langle \alpha \,|\beta\rangle \right|^2 = \left| e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+\alpha\beta^*} \right|^2 = e^{-|\alpha-\beta|^2}.$$

Therefore, they form an over-complete basis:

$$\frac{1}{\pi}\int |\alpha\rangle \langle\alpha|\, d^2\alpha = \mathbb{1}.$$

Coherent States

The displacement operator[3] can also be defined by requiring that $\hat{D}$ is a function of a complex parameter α and displaces $\hat{a}$ according to

$$\hat{D}^{-1}(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha, \qquad \hat{D}^{-1}(\alpha)\hat{a}^{\dagger}\hat{D}(\alpha) = \hat{a}^{\dagger} + \alpha^{*}.$$

With the help of an arbitrary coherent state $\left|\beta\right\rangle$, it can be proven that

$$\hat{a}\hat{D}(\alpha)\left|\beta\right\rangle = (\alpha + \beta)\hat{D}(\alpha)\left|\beta\right\rangle, \quad \hat{D}(-\alpha)\left|\alpha\right\rangle = \left|0\right\rangle$$

One explicit form of this relation is that $\hat{D}(\alpha)\left|0\right\rangle = \left|\alpha\right\rangle$ following which $\hat{D}(d\alpha)$ can be expressed to the first order as $\hat{D}(d\alpha) = 1 + \hat{a}^{\dagger}d\alpha - \hat{a}d\alpha^{*}$ in order to satisfy the relations derived above. We consider increments of α of the form $d\alpha = \alpha d\lambda$ where λ is a real parameter. Then if we also assume the operators $\hat{D}$ have the group multiplication property $\hat{D}(\alpha(\lambda + d\lambda)) = \hat{D}(\alpha d\lambda)\hat{D}(\alpha\lambda)$, we can solve for the differential equation

$$\frac{d}{d\lambda}\hat{D}(\alpha\lambda) = (\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a})\hat{D}(\alpha\lambda), \qquad \hat{D}(\alpha) \overset{\lambda=1}{=} e^{\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}}$$

[3] Glauber, Roy J. "Coherent and incoherent states of the radiation field". *Physical Review* 131.6 (1963): 2766.APA

Coherent States

The displacement operator can thus be defined as $\hat{D}(\alpha) \equiv e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$ with the following properties

Properties of the displacement operator

$$|\alpha\rangle = \hat{D}(\alpha)\,|0\rangle\,, \hat{D}(\alpha)\hat{D}(\beta) = e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)}\hat{D}(\alpha+\beta)$$

$$\hat{D}^\dagger(\alpha) = \hat{D}^{-1}(\alpha) = \hat{D}(-\alpha)$$

A useful theorem for operator calculations

Baker-Campbell-Hausdorff formula

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-[\hat{A},\hat{B}]/2}e^{(2[\hat{B},[\hat{A},\hat{B}]]+[\hat{A},[\hat{A},\hat{B}]])/6}\ldots$$

which gives the other expression of the displacement operator

$$D(\alpha) = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}.$$

Properties of Coherent States

Recall the time evolution of the Fock states:

Fock state evolution

$$|1,t\rangle = |1\rangle\, e^{-i\omega t} \quad or \quad |n,t\rangle = |n\rangle\, e^{-in\omega t}$$

This property is useful in the context of quantum metrology[a][b], since it shows that $|n\rangle$ state exhibits a **de Broglie wavelength of $\boldsymbol{\lambda/n}$** where λ is the vacuum wavelength.

[a] Mitchell, Morgan W., Jeff S. Lundeen, and Aephraem M. Steinberg. "Super-resolving phase measurements with a multiphoton entangled state." *Nature* 429.6988 (2004): 161.

[b] Walther, Philip, et al. "De Broglie wavelength of a non-local four-photon state." *Nature* 429.6988 (2004): 158.

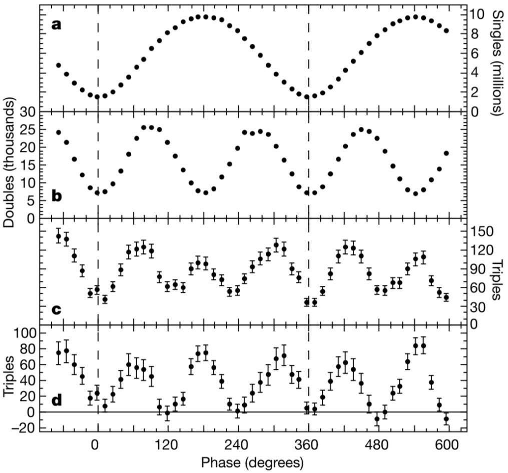
It then follows for a coherent state:

Coherent state evolution

$$|\alpha,t\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}\underbrace{|n\rangle\, e^{-in\omega t}}_{|n,t\rangle} = e^{-|\alpha|^2/2}\sum_n \frac{1}{\sqrt{n!}}\left(\alpha e^{-i\omega t}\right)^n |n\rangle = \left|\alpha e^{\boldsymbol{-i\omega t}}\right\rangle$$

despite the individual Fock state components evolving at $e^{-in\omega t}$.

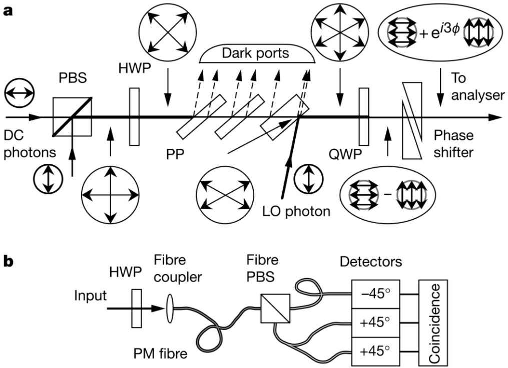
Dynamics of Fock states[4]



Super-resolving phase measurements with a multiphoton entangled state

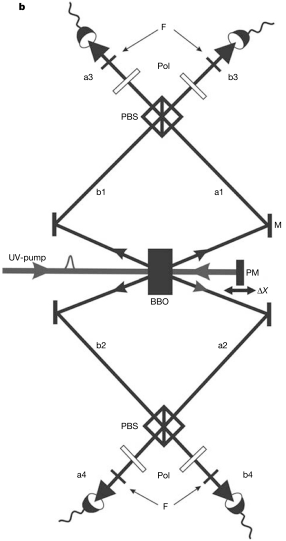
M. W. Mitchell, J. S. Lundeen & A. M. Steinberg

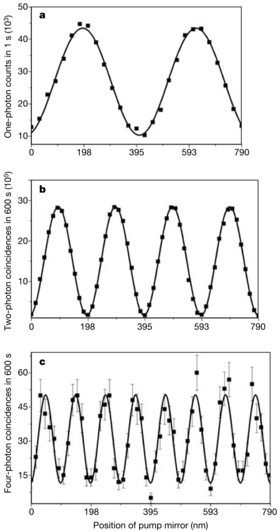
Department of Physics, University of Toronto, 60 St George Street, Toronto, Ontario M5S 1A7, Canada



[4] Mitchell, Morgan W., Jeff S. Lundeen, and Aephraem M. Steinberg. "Super-resolving phase measurements with a multiphoton entangled state." *Nature* 429.6988 (2004): 161.

Dynamics of Fock states[5]





De Broglie wavelength of a non-local four-photon state

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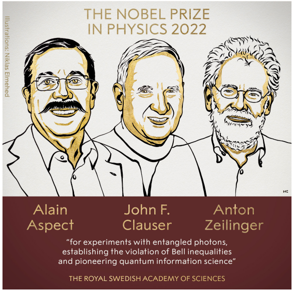


[5]Walther, Philip, et al. "De Broglie wavelength of a non-local four-photon state." *Nature* 429.6988 (2004): 158.

Properties of Coherent States

Normally ordered operators

$$:\hat{a}^\dagger \hat{a} \hat{a}^\dagger: \;\widehat{=}\; \hat{a}^\dagger \hat{a}^\dagger \hat{a}$$

note that

$$\begin{aligned}\langle\alpha| \hat{a}^\dagger \hat{a} \hat{a}^\dagger |\beta\rangle &= \langle\alpha| \hat{a}^\dagger (\hat{a}^\dagger \hat{a} + 1) |\beta\rangle \\ &= \langle\alpha| \underbrace{\hat{a}^\dagger \hat{a}^\dagger \hat{a}}_{\text{normally ordered}} + \hat{a}^\dagger |\beta\rangle = (\alpha^* \alpha^* \beta + \alpha^*) \langle\alpha|\beta\rangle .\end{aligned}$$

Normal ordering is important e.g. for expressing photodetection current operator $\langle \hat{I} \rangle \propto \langle \hat{a}^\dagger \hat{a} \rangle$. [6]
Generally, while a non-empty product of creation and annihilation operators $\hat{O}$ may satisfy

$$\langle 0| \hat{O} |0\rangle \neq 0,$$

the normal ordered version of it $:\hat{O}:$ always satisfies

$$\langle 0| :\hat{O}: |0\rangle = 0.$$

[6] An otherwise ordered current operator would yield $\langle 0| \hat{I} |0\rangle \propto \langle 0| \hat{a} \hat{a}^\dagger |0\rangle = \langle 0| \hat{a}^\dagger \hat{a} + 1 |0\rangle = 1$ for the vacuum state of a field!

Properties of Coherent States

Statistical properties

$$P_\alpha(n) = |\langle n|\alpha\rangle|^2 = \left| e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}$$

Therefore the photon number distribution of a coherent state obeys the Poisson distribution.

$$\langle \hat{n} \rangle = \left\langle \hat{a}^\dagger \hat{a} \right\rangle \hat{=} \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = |\alpha|^2$$

$$\begin{aligned} \left\langle \hat{n}^2 \right\rangle &= \langle \alpha | \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} | \alpha \rangle = \langle \alpha | \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} + \hat{a}^\dagger \hat{a} | \alpha \rangle \\ &= |\alpha|^4 + |\alpha|^2 \end{aligned}$$

Therefore,

$$\Delta \hat{n}^2 \equiv \left\langle (\hat{n} - \langle \hat{n} \rangle)^2 \right\rangle = \left\langle \hat{n}^2 \right\rangle - \langle \hat{n} \rangle^2 = |\alpha|^2 = \langle \hat{n} \rangle$$

Properties of Coherent States

Recall that the wave function of the ground state $|0\rangle$ of a particle with mass $m = 1$ is $\psi_0(x) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{\omega}{2\hbar}x^2}$, which is a minimum uncertainty state. Following a procedure similar to how we obtained $\psi_0(x)$, we can get the wave function of a coherent state:[7]

Coherent state wave function

$$\psi(x, t = 0) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{\omega}{2\hbar}(x-A)^2}$$

and the time evolution is given by:

$$|\psi(x, t)|^2 = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{\omega}{\hbar}(x-A\cos\omega t)^2}.$$

This evolution describes a harmonic oscillation of the wavepacket displacement around 0 with amplitude A.

[7] Scully, M.O., Zubairy, M.S. "Quantum optics" (1999). Chapter 2

Properties of Coherent States

Although coherent state has a nonzero average photon occupancy $\langle \hat{n} \rangle = |\alpha|^2$, it remains a minimum uncertainty state, as

Uncertainty relation for a coherent state

$$\Delta\hat{x} \cdot \Delta\hat{p} = \frac{\hbar}{2}$$

where $\Delta\hat{x}^2 \equiv \left\langle (\hat{x} - \langle \hat{x} \rangle)^2 \right\rangle$ and $\Delta\hat{p}^2 \equiv \left\langle (\hat{p} - \langle \hat{p} \rangle)^2 \right\rangle$ with

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \quad \text{and} \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a}).$$

This is particularly intuitive as coherent state is simply a displaced ground state.

Squeezed States

Quadratures[8]

We define quadratures $\hat{X}_1$, $\hat{X}_2$ as

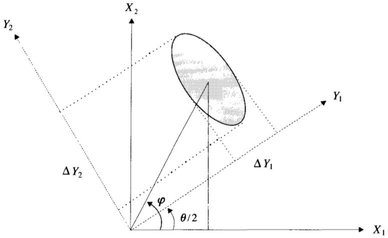
$$\hat{X}_1 = \frac{\hat{a} + \hat{a}^\dagger}{2} \quad \text{and} \quad \hat{X}_2 = \frac{\hat{a} - \hat{a}^\dagger}{2i}$$

with $[\hat{X}_1, \hat{X}_2] = \frac{i}{2}$. For a coherent state $\Delta\hat{X}_1^2 = \Delta\hat{X}_2^2 = \frac{1}{4}$ such that $\Delta\hat{X}_1^2 \cdot \Delta\hat{X}_2^2 = \frac{1}{16}$ which corresponds to the minimum uncertainty.

The quadratures can be visualized in the optical phase space (cf. figure). The quadrature at an arbitrary angle can also be defined as:

$$\hat{Y}_1 + i\hat{Y}_2 = (\hat{X}_1 + i\hat{X}_2)e^{-i\theta/2} = \hat{a}e^{-i\theta/2},$$

where $\theta/2$ is defined as the rotation angle of the quadrature basis. Generally **we use** $-\theta/2 = i\omega t$ **to cancel the time evolution**.



[8] Scully, M.O., Zubairy, M.S. "Quantum optics" (1999). Chapter 2

Squeezed States

Squeezing operator

$$\hat{S}(\xi) = e^{\frac{1}{2}\xi^*\hat{a}^2 - \frac{1}{2}\xi\hat{a}^{\dagger 2}}$$

where $\xi = re^{i\theta}$ is an arbitrary complex number[a]. It has some useful unitary transformation properties[b]

$$\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) = \hat{a}\cosh r - \hat{a}^\dagger e^{i\theta}\sinh r$$
$$\hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi) = \hat{a}^\dagger\cosh r - \hat{a}e^{-i\theta}\sinh r.$$

It's then straight forward to see its operation on quadrature operators $\hat{Y}_i^{(\theta)}$

$$\hat{S}^\dagger(\xi)(\hat{Y}_1 + i\hat{Y}_2)\hat{S}(\xi) = \hat{Y}_1 e^{-r} + i\hat{Y}_2 e^{r}$$

which maintains the minimum uncertainty as

$$\Delta\hat{Y}_1^2 = \frac{1}{4}e^{-2r} \quad \text{and} \quad \Delta\hat{Y}_2^2 = \frac{1}{4}e^{2r}$$

[a] Some books also use the 2θ convention instead of θ.

[b] It is useful to define $u = \cosh r$ and $v = e^{i\theta}\sinh r$ to simplify the expressions. Refer to slides 22 23 for derivation.

Squeezed States

Squeezed coherent state

$$|\alpha,\xi\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle$$

with the statistical properties

$$\langle\alpha,\xi|\,\hat{n}\,|\alpha,\xi\rangle = \langle\hat{n}\rangle = \left\langle\hat{a}^\dagger\hat{a}\right\rangle = |\alpha|^2 + \sinh^2 r$$

$$\langle\alpha,\xi|\,(\hat{n}-\langle\hat{n}\rangle)^2\,|\alpha,\xi\rangle = |\alpha\cosh r - \alpha^* e^{i\theta}\sinh r|^2 + 2\cosh^2 r\sinh^2 r$$

Notice that for a **squeezed** vacuum state $|0,\xi\rangle$ the mean photon occupancy is **non-zero** as only the $|0\rangle$ is the quantum ground state.

Note that the operators $\hat{D}(\alpha)$ and $\hat{S}(\xi)$ are noncommuting, but they follow the rule

$$\hat{D}(\alpha)\hat{S}(\xi) = \hat{S}(\xi)\hat{D}(\beta) \qquad \beta = \alpha\cosh r + \alpha^* e^{i\theta}\sinh r.$$

There are alternative definitions in different books e.g. some use $|\alpha,\xi\rangle = \hat{S}(\xi)\hat{D}(\alpha)|0\rangle$ which is a less intuitive one.

Two Photon Coherent States[9]

Bogoliubov mode

Consider the operator $\hat{\boldsymbol{b}} = \boldsymbol{\mu}\hat{\boldsymbol{a}} + \boldsymbol{\nu}\hat{\boldsymbol{a}}^{\dagger}$ where $|\mu|^2 - |\nu|^2 = 1$ or equivalently $\mu = \cosh r$ and $\nu = e^{i\theta}\sinh r$. Then $\hat{b}$ obeys the commutation relation $[\hat{b}, \hat{b}^{\dagger}] = 1$ and it has eigenstates (Bogoliubov mode)

$$\hat{b}\left|\beta\right\rangle_g = \beta\left|\beta\right\rangle_g = \hat{D}(\alpha)\hat{S}(\xi)\left|0\right\rangle$$

which are actually a set of particular squeezed coherent states or in other words the two-photon coherent states with relation $\alpha = \mu\beta - \nu\beta^*$. Note that under this relation between α and β, we have an equivalent definition

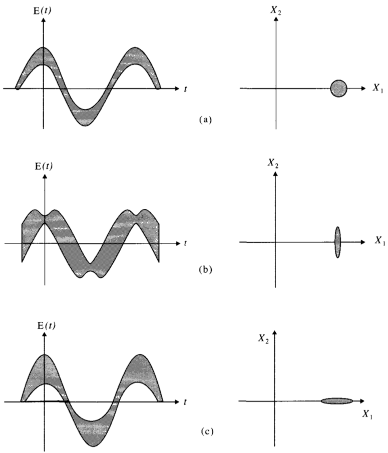
$$\hat{b}\left|\beta\right\rangle_g = \hat{S}(\xi)\hat{D}(\beta)\left|0\right\rangle.$$

This state is referred as the two-photon coherent states because the squeezing operation involved here is a photon-pair operation.

[9] Walls D. F., Milburn G. J. - Quantum Optics - Chapter 2

Two Photon Coherent States[10,11]

Fig. 2.7
Error contours and the corresponding graphs of electric field versus time for (a) a coherent state, (b) a squeezed state with reduced nosie in X_1, and (c) a squeezed state with reduced noise in X_2. (From C. Caves, *Phys. Rev. D* **23**, 1693 (1981).)



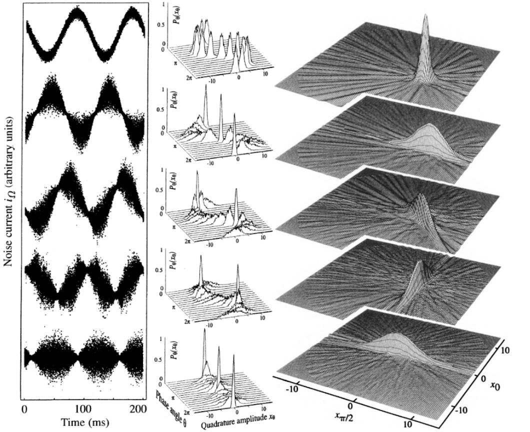


Figure 2 Noise traces in $i_\Omega(t)$ (left), quadrature distributions $P_\phi(x_\phi)$ (centre), and reconstructed Wigner functions (right) of generated quantum states. From the top: Coherent state, phase-squeezed state, state squeezed in the $\phi = 48°$-quadrature, amplitude-squeezed state, squeezed vacuum state. The noise traces as a function of time show the electric fields' oscillation in a 4π interval for the upper four states, whereas for the squeezed vacuum (belonging to a different set of measurements) a 3π interval is shown. The quadrature distributions (centre) can be interpreted as the time evolution of wave packets (position probability densities) during one oscillation period. For the reconstruction of the quantum states a π interval suffices.

[10] Scully, M.O., Zubairy, M.S. "Quantum optics" (1999). Chapter 2

[11] Breitenbach, G., Schiller, S. & Mlynek, J. Measurement of the quantum states of squeezed light. Nature 387, 471–475 (1997).

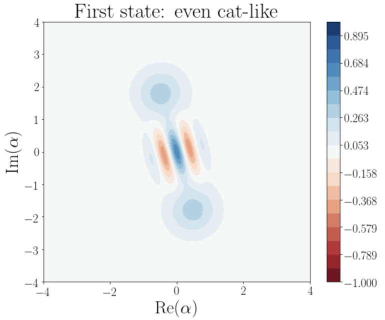
QuTiP[12]

QuTiP

Quantum Toolbox in Python

QuTiP is an open-source software for simulating the dynamics of open quantum system. QuTiP aims to provide user-friendly and efficient numerical simulations of a wide variety of Hamiltonians, including those with arbitrary time-dependence, commonly found in a wide range of physics applications such as quantum optics, trapped ions, superconducting circuits, and quantum nanomechanical resonators.

In [8]:
```
W_even=np.around(W_even, decimals=2)
plt.figure(figsize=(10, 8))
plt.contourf(xvec,xvec, W_even, cmap='RdBu', levels=np.linspace(-1,
 1, 20))
plt.colorbar()
plt.xlabel(r'Re$(\alpha)$', fontsize=label_size), width="300"
plt.ylabel(r'Im$(\alpha)$', fontsize=label_size)
plt.title("First state: even cat-like", fontsize=title_font)
plt.show()
```



[12] Refer to http://qutip.org

QuTiP[13]

In a finite Hilbert space of Fock states one can

define operators and immediatly obtain their matrix form,

e.g. annihilation operator $\hat{a}$

```
a = destroy(5)
```

creation operator $\hat{a}^\dagger$

```
a.dag()
```

Quantum object: dims = [[5], [5]], shape = (5, 5), type = oper, isherm = False

$$\begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.414 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.732 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \end{pmatrix}$$

and compute the commutators $[\hat{a}, \hat{a}^\dagger]$

```
commutator(a, a.dag())
```

Quantum object: dims = [[5], [5]], shape = (5, 5), type = oper, isherm = True

$$\begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.000 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -4.0 \end{pmatrix}$$

A number operator $\hat{a}^\dagger \hat{a} = |n\rangle \langle n|$

in Hilbert space $|0\rangle, \ldots, |99\rangle$

```
a = destroy(100)
n = a.dag()*a
n
```

Quantum object: dims = [[100], [100]], shape = (100, 100), type = oper, isherm = True

$$\begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 2.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 3.000 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 4.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 95.000 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 96.000 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 97.000 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 98.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 99.0 \end{pmatrix}$$

and its expectation value in a coherent state

```
alpha = coherent(100, 2 + 3 * 1j)
print(expect(n, alpha))
```

```
12.999999999999998
```

[13] Refer to http://qutip.org

Squeezed States*

Squeezing operator

One useful equation

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2}[\hat{A},[\hat{A},\hat{B}]] + \ldots \frac{1}{n!}[\overbrace{\hat{A},[\hat{A},\ldots[\hat{A}}^{\text{n } \hat{A}s},\hat{B}]\ldots] + \ldots$$

which can be applied to $S^\dagger(\xi)\hat{a}S(\xi)$ as

$$\begin{aligned} S^\dagger(\xi)\hat{a}S(\xi) &= \hat{a} - \xi\hat{a}^\dagger + \frac{1}{2!}|\xi^2|\hat{a} - \frac{1}{3!}|\xi|^2\xi\hat{a}^\dagger + \ldots \\ &= \hat{a}(1 + \frac{r^2}{2!} + \frac{r^4}{4!} + \ldots) - \hat{a}^\dagger e^{i\theta}(r + \frac{1}{3!}r^3 + \frac{1}{5!}r^5 + \ldots) \\ &= \hat{a}\cosh r - \hat{a}^\dagger e^{i\theta}\sinh r \end{aligned}$$

Squeezed States*

Squeezing operator

To prove the first equation we used, we need to define an operator function $\hat{F}(x) = e^{x\hat{A}}\hat{B}e^{-x\hat{A}} = \sum_0^\infty \frac{1}{n!}\hat{F}_n x^n$ so that we can further derive

$$\frac{d}{dx}\hat{F}(x) = [\hat{A}, \hat{F}(x)].$$

By plugging the definition into this relation we obtain

$$\sum_1^\infty \hat{F}_n \frac{1}{(n-1)!}x^{n-1} = \sum_0^\infty \frac{1}{n!}[\hat{A}, \hat{F}_n]x^n$$

from which we find $\hat{F}_{n+1} = [\hat{A}, \hat{F}_n]$. Since it's easy to find $\hat{F}_0 = \hat{B}$, we have

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{F}(1) = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2}[\hat{A}, [\hat{A}, \hat{B}]] + \ldots \frac{1}{n!}[\overbrace{\hat{A}, [\hat{A}, \ldots [\hat{A}}^{\text{n } \hat{A}s}, \hat{B}] \ldots] + \ldots$$