Quantum Electrodynamics and Quantum Optics: Lecture 14

Fall 2022

Back-action evasion and quantum non-demolition [1]

$$\hat{H} = \hat{H}_{\rm sys} + \hat{H}_{\rm meter} + \hat{H}_{\rm int}$$

Equation of motion for system and meter operators $\hat{A}_{\rm s}$, $\hat{A}_{\rm m}$:

$$i\hbar\partial_t \hat{A}_{\rm s} = [\hat{A}_{\rm s}, \hat{H}_{\rm sys} + \hat{H}_{\rm int}]$$

$$i\hbar\partial_t \hat{A}_{\rm m} = [\hat{A}_{\rm m}, \hat{H}_{\rm meter} + \hat{H}_{\rm int}]$$

Quantum non-demolition (QND) measurement

Repeated measurements of system yields the same results, i.e. no back-action on the system.

[1] Quantum Optics, Scully, Chapter 19

Formal definition of the QND measurement[1]

- Since $\hat{A}_{\mathrm{s}}$ is measured, $\hat{H}_{\mathrm{int}}$ must depend on $\hat{A}_{\mathrm{s}}$.
- Meter cannot be a constant of motion. Hence: $[\hat{A}_{\mathrm{m}}, \hat{H}_{\mathrm{int}}] \neq 0$
- Observable $\hat{A}_{\mathrm{s}}$ should not be affected by coupling during measurement (i.e. back-action evading): $[\hat{A}_{\mathrm{s}}, \hat{H}_{\mathrm{int}}] = 0$ (Eigenstate of $\hat{A}_{\mathrm{s}}$ is eigenstate of $\hat{H}_{\mathrm{int}}$.)
- The unperturbed Hamiltonian $\hat{H}_{\mathrm{s}}$ is not a function of $\hat{A}_{\mathrm{s}}^{\mathrm{conj}}$ (Conjugate variable to $\hat{A}_{\mathrm{s}}$).

$$\frac{\partial \hat{H}_{\mathrm{s}}}{\partial \hat{A}_{\mathrm{s}}} = 0$$

[1] Quantum Optics, Scully, Chapter 19

Quantum non-demolition [1]

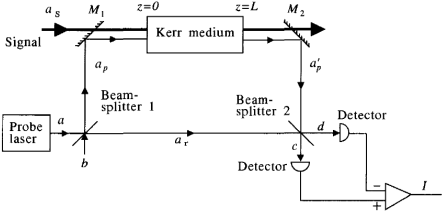

Figure: QND measurement in Kerr medium

$$\hat{H} = \hat{H}_{\text{sys}} + \hat{H}_{\text{meter}} + \hat{H}_{\text{int}} =$$

$$\hbar\omega_{\text{s}}(\hat{a}^{\dagger}{}_{\text{s}}\hat{a}_{\text{s}} + \frac{1}{2}) + \hbar\omega_{\text{m}}(\hat{a}^{\dagger}{}_{\text{m}}\hat{a}_{\text{m}} + \frac{1}{2}) + \underbrace{\kappa\hbar\hat{a}^{\dagger}{}_{\text{s}}\hat{a}_{\text{s}}\hat{a}^{\dagger}{}_{\text{m}}\hat{a}_{\text{m}}}_{\text{Kerr Hamiltonian}}$$

The QND observable in this scheme is $\hat{A}_{\text{s}} = \hat{a}^{\dagger}{}_{\text{s}}\hat{a}_{\text{s}}$, where κ is the Kerr coefficient. (Homodyne phase is the readout.)

[1] Quantum Optics, Scully, Chapter 19

QND measurement[1]

$$\partial_t \hat{a}_{\mathrm{p}} = -i\kappa \hat{A}_{\mathrm{s}} \hat{a}_{\mathrm{p}}$$

$$t = \frac{z}{v}, \partial_z \hat{a}_{\mathrm{p}} = \frac{i\kappa}{v} \hat{A}_{\mathrm{s}} \hat{a}_{\mathrm{p}}$$

$$\hat{a}(L) = \exp\left(\frac{i\kappa}{v} \hat{A}_{\mathrm{s}} \cdot L\right) \hat{a}(0), (\hat{A}_{\mathrm{s}} \text{is constant of motion})$$

Output to homodyne:

$$\hat{c} = \sqrt{T_2} \hat{a}_{\mathrm{p}'} + i\sqrt{1 - T_2} \hat{a}_{\mathrm{r}}, \hat{a} = \sqrt{T_2} \hat{a}_{\mathrm{r}} + i\sqrt{1 - T_2} \hat{a}_{\mathrm{p}'}$$

$$\hat{a}_{\mathrm{p}} = \sqrt{T_1} \hat{b} + i\sqrt{1 - T_1} \hat{a}, \hat{a}_{\mathrm{r}} = \sqrt{T_1} \hat{a} + i\sqrt{1 - T_1} \hat{b}$$

The photo-current difference is $n_{cd} = \hat{c}^\dagger \hat{c} - \hat{d}^\dagger \hat{d}$ for $T_2 = 1/2$ and input b in vacuum $|0\rangle$:

$$\langle n_{cd} \rangle \cong 2\sqrt{T_1(1 - T_1)} \langle \hat{a}^\dagger{}_{\mathrm{s}} \hat{a}_{\mathrm{s}} \rangle \frac{KL}{v} \langle \hat{A}_{\mathrm{s}} \rangle$$

[1] Quantum Optics, Scully, Chapter 19

Back-action evasion measurement[2]

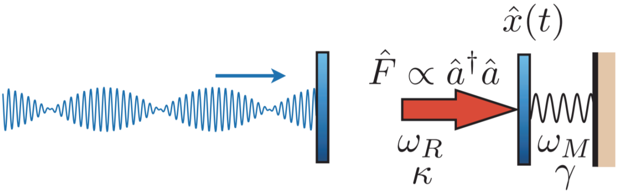

Figure: Back-action evasion measurement in optomechanics

Optomechanical hamiltonian

$$\hat{H}_{\text{int}} = \hbar \hat{a}^\dagger \hat{a} (\hat{b}^\dagger + \hat{b}) x_{\text{ZPF}} \frac{\omega_c}{L}$$

Transforming operators ($\hat{H}_{\text{int}} \rightarrow \hat{\tilde{H}}_{\text{int}}$):

$$\hat{a} \rightarrow \hat{a} e^{i\omega t} \ , \ \hat{b} \rightarrow \hat{b} e^{i\Omega_{\text{m}} t}$$

$$\hat{X}_1 = \hat{b} e^{+i\Omega_{\text{m}} t} + \hat{b}^\dagger e^{-i\Omega_{\text{m}} t} \ , \ \hat{X}_2 = -\hat{b}^\dagger e^{+i\Omega_{\text{m}} t} + \hat{b} e^{-i\Omega_{\text{m}} t}$$

[2] Clerk, A. A., Florian Marquardt, and K. Jacobs. "Back-action evasion and squeezing of a mechanical resonator using a cavity detector." New Journal of Physics 10.9 (2008): 095010.

BAE measurement

$$\hat{\tilde{H}}_{\text{int}} = \hbar \hat{a}^\dagger \hat{a} (\hat{X}_1 \cos(\Omega_{\text{m}} t) + i \hat{X}_2 \sin(\Omega_{\text{m}} t)) \frac{\omega}{L}$$

Note: In the rotating frame ($\omega = \Omega_m$), quadratures are constants of motion unlike $\hat{x}, \hat{p}$. This is due to the fact that they are not coupled. Next consider how to remove the dependence to both quadratures:

$$\hat{a}^\dagger \hat{a} \propto \hat{a}^\dagger \hat{a} \cos^2(\Omega_{\text{m}} t) \text{ , } \hat{a} \to 2\alpha \cos(\Omega_{\text{m}} t) + \delta \hat{a} \quad \text{(Two-tone driving)}$$

After linearizing the Hamiltonian and neglecting DC and second order terms we find:

$$\hat{\tilde{H}}_{\text{int}} = 2\hbar\alpha \underbrace{(\cos(\Omega_{\text{m}} t))}_{\frac{1}{2}(e^{+i\Omega_{\text{m}} t} + e^{-i\Omega_{\text{m}} t})} (\delta\hat{a} + \delta\hat{a}^\dagger)(\hat{b} e^{+i\Omega_{\text{m}} t} + \hat{b}^\dagger e^{-i\Omega_{\text{m}} t}) \frac{\omega}{L}$$

After RWA:

$$\hat{\tilde{H}}_{\text{int}} = 2\hbar\alpha \underbrace{(\delta\hat{a} + \delta\hat{a}^\dagger)}_{\text{optical quadrature}(\hat{X}_L)} (\hat{b} + \hat{b}^\dagger) \frac{\omega}{L} \propto \hat{X}_L \hat{X}_1 \frac{\omega}{L}$$

BAE measurement

In this frame there is no oscillation at $\Omega_{\rm m}$. This removes coupling of $\hat{X}_1$ and $\hat{X}_2$.

Full Hamiltonian

$$\hat{H} = 2\hbar\alpha\frac{\omega}{L}\hat{X}_L(\hat{X}_1(1+\cos(2\Omega_{\rm m}t)) + \hat{X}_2\sin(2\Omega_{\rm m}t))$$

Quadratures equation of motion $g = \alpha\frac{\omega}{L}$:

$$\frac{d}{dt}\hat{X}_1 = -\frac{\Gamma_{\rm m}}{2}\hat{X}_1 + \sqrt{\Gamma_{\rm m}}\hat{X}_1^{\rm in} + 2g\hat{X}_L\sin(2\Omega_{\rm m}t)$$

$$\frac{d}{dt}\hat{X}_2 = -\frac{\Gamma_{\rm m}}{2}\hat{X}_2 + \sqrt{\Gamma_{\rm m}}\hat{X}_2^{\rm in} - 2g\hat{X}_L(1+\cos(2\Omega_{\rm m}t))$$

BAE measurement

Equations can be solved in Fourier domain:

$$\hat{X}_1(\omega) = \chi_a(\omega)(\sqrt{\Gamma_{\mathrm{m}}}\hat{X}_1^{\mathrm{in}} + ig(\hat{X}_L(\omega - 2\Omega_{\mathrm{m}}) - \hat{X}_L(\omega + 2\Omega_{\mathrm{m}})))$$

$$\hat{X}_2(\omega) = \chi_a(\omega)(\sqrt{\Gamma_{\mathrm{m}}}\hat{X}_2^{\mathrm{in}} - ig(\hat{X}_L(\omega) + \hat{X}_L(\omega - 2\Omega_{\mathrm{m}}) + \hat{X}_L(\omega + 2\Omega_{\mathrm{m}})))$$

Where $\chi_a(\omega) = \frac{1}{i\omega - \Gamma_{\mathrm{m}}/2}$ is quadrature susceptibility.

Spectrum at the mechanical resonance ($\omega = 0$)

$$S_{\hat{X}_1\hat{X}_1}(\omega = 0) = \frac{4}{\Gamma_{\mathrm{m}}}(\bar{n}_{\mathrm{th}} + \frac{1}{2} + \frac{C}{1 + 16\frac{\Omega_{\mathrm{m}}^2}{\kappa^2}})$$

$$S_{\hat{X}_2\hat{X}_2}(\omega = 0) = \frac{4}{\Gamma_{\mathrm{m}}}(\bar{n}_{\mathrm{th}} + \frac{1}{2} + 2\frac{C}{1 + 1/(1 + 16\frac{\Omega_{\mathrm{m}}^2}{\kappa^2})})$$

So back-action only appears in $S_{\hat{X}_2\hat{X}_2}$ not in $S_{\hat{X}_1\hat{X}_1}$ (BAE measurement)