Quantum Electrodynamics and Quantum Optics: Lecture 11

Fall 2025

Stern-Gerlach experiment[1]

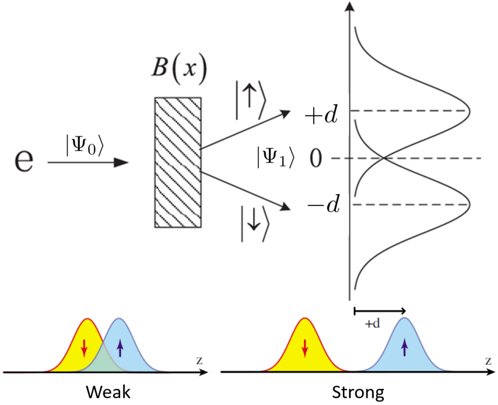
Consider two spin states $(|\uparrow\rangle, |\downarrow\rangle)$. Using magnetic field gradient we apply different forces on them: $F \propto (\nabla \cdot B) \cdot \hat{\sigma}_z$.

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \otimes |x_0\rangle$$

After evolution by magnetic field:

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle |x_+\rangle + |\downarrow\rangle |x_-\rangle)$$

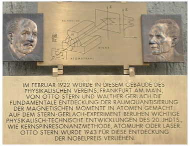
Thus we have entangled the spin with the motional degree of freedom.



[1] Clerk, Aashish A., et al. "Introduction to quantum noise, measurement, and amplification." Reviews of Modern Physics 82.2 (2010): 1155.

Stern-Gerlach experiment

If the spread of $|\psi_{\pm}(x)|^2$ gets bigger than the width of wave packets, we will have a strong projective measurement.



Strong measurement decoheres the system

Consider a strong measurement, where the initial state is

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle + |\uparrow\rangle)|x_0\rangle$$

is an eigenstate of $\hat{\sigma}_x$, so that $\langle\Psi_0|\,\hat{\sigma}_x\,|\Psi_0\rangle = 1$. This expectation value is a measure of the coherence of the state. After a measurement the state becomes

$$\langle\Psi_1|\,\hat{\sigma}_x\,|\Psi_1\rangle = \frac{1}{\sqrt{2}}(\langle x_-|x_+\rangle + \langle x_+|x_-\rangle),$$

evidently $\langle x_-|x_+\rangle \to 0$ for a strong projective measurements. Thus measurement induces decoherence of a state.

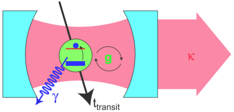
Cavity QED in the dispersive limit

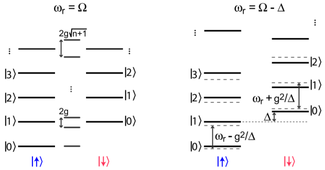
Two-level system in a cavity

$$\hat{H} = \hbar\left(\omega_c + \hat{\sigma}_z \frac{g^2}{\Delta}\right)\hat{a}^\dagger\hat{a} + \hbar\left(\omega_{eg} + \frac{g^2}{\Delta}\right)\frac{\hat{\sigma}_z}{2}$$

$$E_n^{\pm} = \hbar\omega_c(n+1) \pm \hbar\left(\frac{\omega_{eg}}{2} \pm \frac{\Omega_n^2}{4\Delta}\right)$$

where $\Omega_n = 2g\sqrt{n+1}$, $\Delta = \omega_{eg} - \omega_c$

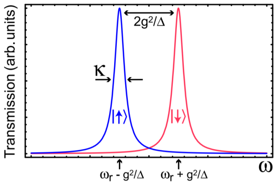




Thus transition frequencies are

$$\tilde{\omega}_{eg} = \frac{1}{\hbar}(E_n^+ - E_{n-1}^-) = \omega_{eg} + (2n+1)\frac{g^2}{\Delta}$$

$$\tilde{\omega}_c = \frac{1}{\hbar}(E_n^- - E_{n-1}^-) = \omega_c - \frac{g^2}{\Delta}$$



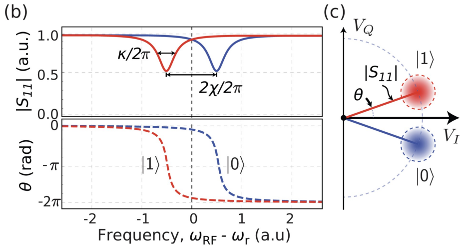
Dispersive measurement of a two-level system in the cavity

Response of the dressed cavity

$$\hat{a}_{\text{out}} + \hat{a}_{\text{in}} = \sqrt{\kappa}\hat{a}$$

$$\frac{d}{dt}\hat{a} = -i(\omega_c + \hat{\sigma}_z\frac{g^2}{\Delta})\hat{a} - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_{\text{in}}$$

$$\frac{\langle\hat{a}_{\text{out}}\rangle}{\langle\hat{a}_{\text{in}}\rangle} = r(\omega_c) = -\left(\frac{1 + 2i\frac{g^2}{\Delta\kappa}\langle\hat{\sigma}_z\rangle}{1 - 2i\frac{g^2}{\Delta\kappa}\langle\hat{\sigma}_z\rangle}\right) = |r|e^{i\phi_0}$$



Phase response at the cavity frequency

$$\phi_0 \approx 4\frac{g^2}{\Delta}\frac{1}{\kappa}\langle\hat{\sigma}_z\rangle$$

Phase-number uncertainty

Considering a coherent state incident on the cavity $|\psi_{\text{in}}\rangle = |\alpha\rangle$, what is the uncertainty in the homodyne detection of the phase of the reflected radiation $|\psi_{\text{out}}\rangle = |re^{i\phi}\alpha\rangle$?

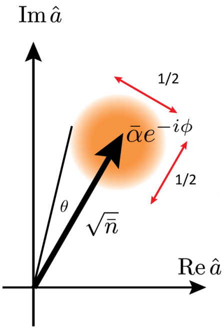
For a coherent state, $\langle \hat{X}_1^2 \rangle^{1/2} = \langle \alpha | \hat{X}_1^2 | \alpha \rangle^{1/2} = 1/2 = \langle \hat{X}_2^2 \rangle^{1/2}$, where $\hat{X}_1 = (\hat{a} + \hat{a}^\dagger)/2$, $\hat{X}_2 = (\hat{a} - \hat{a}^\dagger)/2i$ is the field quadrature operator

Variances

$$\langle \Delta\hat{\phi}^2 \rangle = \frac{1}{2} \frac{\Delta \hat{X}_2^2}{|\alpha|^2} = \frac{1}{4\bar{N}}$$

$$\langle \Delta\hat{N}^2 \rangle = \bar{N}$$

$$\langle \Delta\hat{N}^2 \rangle \langle \Delta\hat{\phi}^2 \rangle = \frac{1}{4}$$



Measurement rate

$$\text{SNR} = \frac{\phi^2}{\langle \Delta\hat{\phi}^2 \rangle} = \frac{\phi^2}{S_{\phi\phi} t^{-1}}$$

where t^{-1} is the measurement bandwidth.

$$\langle \Delta\hat{\phi}^2 \rangle = \frac{1}{4\dot{\mathrm{N}}t} = \frac{\hbar\omega}{4P}\frac{1}{t}$$

where $\dot{\mathrm{N}}$ is the average photon flux.

Measurement rate

$$\Gamma_{\mathrm{m}} \equiv \frac{\text{SNR}}{2t} = \frac{(\phi)^2}{2S_{\phi\phi}}$$

Heisenberg uncertainty for spectral densities

$$S_{\phi\phi} S_{\dot{\mathrm{N}}\dot{\mathrm{N}}} = \frac{1}{4}$$

Measurement back-action

Consider $\hat{H} = \frac{\hbar}{2}(\omega_{\text{eg}} + \frac{g^2}{\Delta}\hat{a}^\dagger\hat{a})\hat{\sigma}_z + \hbar\omega_c\hat{a}^\dagger\hat{a}$ thus photon induced energy level shifts are: $\Delta\omega_{\text{eg}} = \frac{g^2}{\Delta}\hat{a}^\dagger\hat{a}$. Linearizing by $\hat{a} = \bar{a} + \delta\hat{a}$, and equivalently $\hat{n} = \bar{n} + \delta\hat{n}$

$$\Delta\omega_{\text{eg}} = \underbrace{\frac{g^2}{\Delta}\bar{n}}_{\text{mean}} + \underbrace{\frac{g^2}{\Delta}\delta\hat{n}}_{\text{fluctuation}}$$

consider:

$$\frac{d}{dt}\hat{\sigma}^+ = -i\frac{\omega_{\text{eg}}}{2}\hat{\sigma}^+ - i\frac{g^2}{\Delta}\delta\hat{n}\hat{\sigma}^+$$

dephasing $\langle\hat{\sigma}^+(t)\hat{\sigma}^-(0)\rangle = \langle e^{-i\hat{\phi}(t)}\rangle$, $\hat{\phi}(t) = \int_0^t \Delta\omega_{\text{eg}}(t')dt'$

Dephasing

$$\langle\hat{\sigma}^+(t)\hat{\sigma}^-(0)\rangle \approx e^{-\Gamma_\phi t},\ \Gamma_\phi = 4\phi_0^2\frac{\kappa}{2}\bar{n} = 2\phi_0^2 S_{\dot{N}\dot{N}} = \frac{2\phi_0^2}{4S_{\phi\phi}} = \Gamma_{\text{meas}}$$

Hence both measurement rate and dephasing rate are equal.

Dephasing

Dephasing from atom-photon interaction

Consider the initial state $|\Phi_0\rangle = (|g\rangle + |e\rangle)/\sqrt{2}$, the off-diagonal term of the density matrix $\langle\Psi(t)|\,\hat{\sigma}^+\,|\Psi(t)\rangle = e^{-\Gamma_\phi t}$ decays due to entanglement of cavity and atom. Assuming the initial state to be $|\Phi_0\rangle = (|g\rangle + |e\rangle)/\sqrt{2} \otimes |\alpha\rangle$, after the interaction, the state changes to

$$|\Phi(t)\rangle = \frac{1}{\sqrt{2}}(e^{-i\omega_{\mathrm{eg}}t/2}\,|g\rangle \otimes |r_e\alpha\rangle + e^{+i\omega_{\mathrm{eg}}t/2}\,|e\rangle \otimes \left|r_g\alpha\right\rangle),$$

where $r_{e,g} = r(\omega_{\mathrm{c}} \pm i\frac{g^2}{\Delta})$ is the cavity reflection coefficient. After tracing over the light field states,

$$s_{\mathrm{eg}} = \mathrm{Tr}(\langle 1|\,\Psi_t\rangle\langle\Psi_t\,|e\rangle) = e^{i\omega_{\mathrm{eg}}t/2}e^{-|\alpha|^2(1-r_e^* r_g)} = e^{i\omega_{\mathrm{eg}}t/2}e^{-2\phi_0^2\bar{N}},$$

we again find the dephasing factor

$$e^{-\Gamma_\phi t}$$

Paper for next week's presentation

PHYSICAL REVIEW A VOLUME 47, NUMBER 4 APRIL 1993

Quantum limits in interferometric detection of gravitational radiation

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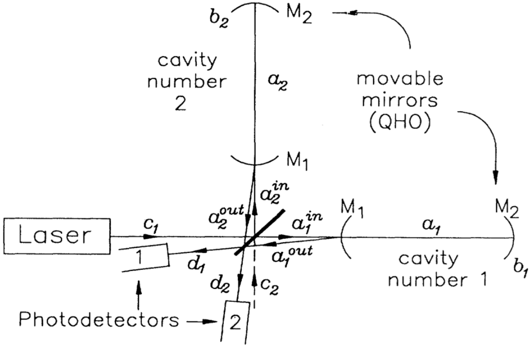

Questions for next week's paper

- What's the phase-relation of gravitational "forces" at two arms?
- Optical phase relation between the two arms, which optical quadratures are detected from the two arms?
- What's the expression for the "signal", and the expression for the "noise"? What are the contributions of the "noise"?
- What's the trade-off that leads to an "optimal" optical power? What's the minimally detectable gravitational displacement? What if the harmonic oscillator is in a thermal state?