Quantum Electrodynamics and Quantum Optics

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Exercise No.8

8.1 Equivalence of rE- and pA-Hamiltonians

We consider a single mode electric field

$$\mathbf{E}(t) = \mathbf{E}_0 \cos(\omega t) \tag{1}$$

The vector potential is given by

$$\mathbf{A}(t) = -\frac{1}{\omega}\mathbf{E}_0 \sin(\omega t) \tag{2}$$

The rE and pA Hamiltonian are then replaced by

$$\hat{H}_{\mathbf{r}\cdot\mathbf{E}} = -\, q\mathbf{r}\cdot\mathbf{E} = -q\mathbf{r}\cdot\mathbf{E}_0\cos(\omega t) \tag{3}$$

$$\hat{H}_{\mathbf{p}\cdot\mathbf{A}} = -\,\frac{q}{m}\mathbf{p}\cdot\mathbf{A} = \frac{q}{m\omega}\mathbf{p}\cdot\mathbf{E}_0\sin(\omega t) \tag{4}$$

Now we calculate the off-diagonal elements of the transition matrix of the two Hamiltonian:

$$\langle f|\hat{H}_{\mathbf{r}\cdot\mathbf{E}}|i\rangle = -q\,\langle f|\mathbf{r}\cdot\mathbf{E}_0|i\rangle\cos(\omega t) \tag{5}$$

$$\begin{aligned}
\langle f|\hat{H}_{\mathbf{p}\cdot\mathbf{A}}|i\rangle &= \langle f|\frac{q}{m\omega}\mathbf{p}\cdot\mathbf{E}_0|i\rangle\sin(\omega t) \\
&= \langle f|\frac{q}{m\omega}(\frac{im}{\hbar})[\hat{H},\mathbf{r}]\cdot\mathbf{E}_0|i\rangle\sin(\omega t) \\
&= \frac{iq}{\omega\hbar}[\,\langle f|\hat{H}\mathbf{r}\cdot\mathbf{E}_0|i\rangle - \langle f|\mathbf{r}\cdot\mathbf{E}_0\hat{H}|i\rangle]\sin(\omega t) \\
&= \frac{iq}{\omega}(\omega_f-\omega_i)\,\langle f|\mathbf{r}\cdot\mathbf{E}_0|i\rangle\sin(\omega t)
\end{aligned} \tag{6}$$

Therefore, the off-diagonal elements are different by only a factor related to the frequency.

Solution: Wigner-Weisskopf theory

The solution can be find in Scully, M.O. and Zubairy, M.S., 1999. Quantum optics, Chapter 6.3.

8.2 Synthesizing arbitrary quantum states

8.2.1 Describe the Hamiltonian

The Ω_{JC} term decribes the interaction between atom('resonator mode') and photons. The Ω_{Rabi} term and the Ω_D terms describes the external drive applied on the resonator and the waveguide, which can be understood per writing down the evolution equation.

8.2.2 State Evolution

Only Rabi Ω_{Rabi}:

$$|\psi(t)\rangle = \cos(1/2\Omega_{Rabi}t)|1,0\rangle + \sin(1/2\Omega_{Rabi}t)|0,0\rangle \tag{7}$$

Only Drive Ω_D:

$$|\psi(t)\rangle = e^{iHt/\hbar}|1,0\rangle = |1,\Omega_D t/2\rangle \;\; \text{(Coherent state)} \tag{8}$$

Only JC-interaction Ω_{JC}:
The equation reads:

$$\begin{aligned}
\partial_t a &= -i\Omega_{JC}/2\, b \\
\partial_t b &= -i\Omega_{JC}/2\, a
\end{aligned} \tag{9}$$

Thus:

$$|\psi(t)\rangle = \cos(1/2\,|\Omega_{JC}|t)|1,0\rangle + \sin(1/2\,|\Omega_{JC}|t)|0,1\rangle \tag{10}$$

8.2.3 Arbitrary state generation

Please refer to *Synthesizing arbitrary quantum states in a superconducting resonator.*

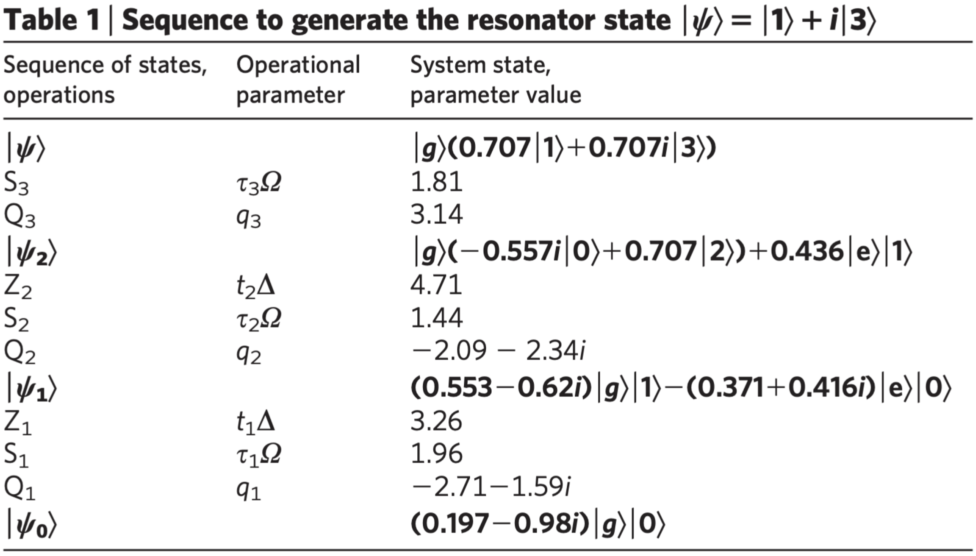

Table 1 | Sequence to generate the resonator state $|\psi\rangle = |1\rangle + i|3\rangle$

Sequence of states, operations	Operational parameter	System state, parameter value
$\|\psi\rangle$		$\|g\rangle(0.707\|1\rangle+0.707i\|3\rangle)$
S_3	$\tau_3\Omega$	1.81
Q_3	q_3	3.14
$\|\psi_2\rangle$		$\|g\rangle(-0.557i\|0\rangle+0.707\|2\rangle)+0.436\|e\rangle\|1\rangle$
Z_2	$t_2\Delta$	4.71
S_2	$\tau_2\Omega$	1.44
Q_2	q_2	$-2.09 - 2.34i$
$\|\psi_1\rangle$		$(0.553-0.62i)\|g\rangle\|1\rangle-(0.371+0.416i)\|e\rangle\|0\rangle$
Z_1	$t_1\Delta$	3.26
S_1	$\tau_1\Omega$	1.96
Q_1	q_1	$-2.71-1.59i$
$\|\psi_0\rangle$		$(0.197-0.98i)\|g\rangle\|0\rangle$

8.3 Solution : Bloch-Siegert shift: An example of non-RWA effect

Here we attempt to give the solution using the Laplace Transform method. Considering the equations for the coherences given by the coupled system

$$\begin{cases} \dot{\rho}_{12} = \left(i\omega_0 - \frac{\Gamma}{2}\right)\rho_{12} + \Gamma\rho_{21}, \\ \dot{\rho}_{21} = \left(-i\omega_0 - \frac{\Gamma}{2}\right)\rho_{21} + \Gamma\rho_{12}, \end{cases} \tag{11}$$

and the intitial conditions $\rho_{12}(0), \rho_{21}(0)$, we perform a Laplace transformation of the system. It is given by

$$\mathcal{L}[f(t)](s) = \int_0^\infty \mathrm{d}t\, f(t) e^{-st}, \tag{12}$$

where the parameter $s = \sigma + i\omega \in \mathbb{C}$. We will use the following properties for $\mathcal{L}$:

$$\begin{cases} \mathcal{L}[cf](s) = c\mathcal{L}[f](s), \text{ with } c \in \mathbb{C} \\ \mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0). \end{cases} \tag{13}$$

The tranformed system is given by

$$\begin{cases} s\mathcal{L}[\rho_{12}](s) - \rho_{12}(0) = \left(i\omega_0 - \frac{\Gamma}{2}\right)\mathcal{L}[\rho_{12}](s) + \Gamma\mathcal{L}[\rho_{21}](s), \\ s\mathcal{L}[\rho_{21}](s) - \rho_{21}(0) = \left(-i\omega_0 - \frac{\Gamma}{2}\right)\mathcal{L}[\rho_{21}](s) + \Gamma\mathcal{L}[\rho_{12}](s), \end{cases} \tag{14}$$

then

$$\begin{cases} \mathcal{L}[\rho_{12}](s) = \frac{1}{s-i\omega_0+\frac{\Gamma}{2}}\left(\Gamma\mathcal{L}[\rho_{21}](s) + \rho_{12}(0)\right) = \frac{1}{s-i\omega_0+\frac{\Gamma}{2}}\left(\Gamma\frac{\Gamma\mathcal{L}[\rho_{12}](s)+\rho_{21}(0)}{s+i\omega_0+\frac{\Gamma}{2}} + \rho_{12}(0)\right), \\ \mathcal{L}[\rho_{21}](s) = \frac{1}{s+i\omega_0+\frac{\Gamma}{2}}\left(\Gamma\mathcal{L}[\rho_{12}](s) + \rho_{21}(0)\right) = \frac{1}{s+i\omega_0+\frac{\Gamma}{2}}\left(\Gamma\frac{\Gamma\mathcal{L}[\rho_{21}](s)+\rho_{12}(0)}{s-i\omega_0+\frac{\Gamma}{2}} + \rho_{21}(0)\right). \end{cases} \tag{15}$$

We can now isolate the Laplace transform of the coherences to obtain

$$\begin{cases} \mathcal{L}[\rho_{12}](s) = \frac{\left(s+\frac{\Gamma}{2}+i\omega_0\right)\rho_{12}(0)+\Gamma\rho_{21}(0)}{\left|s+\frac{\Gamma}{2}+i\omega_0\right|^2-\Gamma^2}, \\ \mathcal{L}[\rho_{21}](s) = \frac{\left(s+\frac{\Gamma}{2}-i\omega_0\right)\rho_{21}(0)+\Gamma\rho_{12}(0)}{\left|s+\frac{\Gamma}{2}-i\omega_0\right|^2-\Gamma^2}. \end{cases} \tag{16}$$

Assuming that $\Gamma \ll \omega_0$ (we will add a term $\propto (\Gamma/\omega_0)^4$), we write the denominator as

$$\begin{aligned}
D(s) &= \left| s + \frac{\Gamma}{2} - i\omega_0 \right|^2 - \Gamma^2 = \left(s + \frac{\Gamma}{2} \right)^2 + \omega_0^2 \left(1 - \frac{\Gamma^2}{\omega_0^2} \right) \\
&= \left(s + \frac{\Gamma}{2} \right)^2 + \omega_0^2 \left(1 - \frac{\Gamma^2}{\omega_0^2} + \left(\frac{\Gamma^2}{2\omega_0^2} \right)^2 \right) = \left(s + \frac{\Gamma}{2} \right)^2 + \left(\omega_0 - \frac{\Gamma^2}{2\omega_0} \right)^2 \\
&= (s - z_+)\,(s - z_-)\,,
\end{aligned} \tag{17}$$

where $z_\pm = -\frac{\Gamma}{2} \pm i \left(\omega_0 - \frac{\Gamma^2}{2\omega_0} \right)$ are the poles of both Laplace transforms. The real part of $z_\pm$ is linked to the damping of the solution, while the imaginary part corresponds to the oscillating part. We see that the frequency changes from $\omega_0 \to \omega_0 - \frac{\Gamma^2}{2\omega_0}$, which is the so-called Bloch-Siegert shift when $\Gamma \ll \omega_0$. To gain a better understanding, we consider the following two Laplace transforms :

$$\begin{cases}
\mathcal{L}[e^{-at}\cos(bt)](s) = \frac{s+a}{(s+a)^2+b^2}, \\
\mathcal{L}[e^{-at}\sin(bt)](s) = \frac{b}{(s+a)^2+b^2},
\end{cases} \tag{18}$$

Recall that the Laplace transform of ρ_{12} (a similar expression was found for ρ_{21}) was found to be

$$\mathcal{L}[\rho_{12}](s) = \frac{(s + \frac{\Gamma}{2})\,\rho_{12}(0) + i\omega_0\rho_{12}(0) + \Gamma\rho_{21}(0)}{(s + \frac{\Gamma}{2})^2 + \left(\omega_0 - \frac{\Gamma^2}{2\omega_0} \right)^2}, \tag{19}$$

from the linearity of the Laplace transform, we deduce that in the time-domain, the expression of the coherence is

$$\rho_{12}(t) = e^{-\Gamma t/2} \left\{ \rho_{12}(0) \cos \left[\left(\omega_0 - \frac{\Gamma^2}{2\omega_0} \right) t \right] + \frac{i\omega_0\rho_{12}(0) + \Gamma\rho_{21}(0)}{\omega_0 - \frac{\Gamma^2}{2\omega_0}} \sin \left[\left(\omega_0 - \frac{\Gamma^2}{2\omega_0} \right) t \right] \right\}. \tag{20}$$

With this final expression, we can clearly observe the decaying part with a rate $\Gamma/2$ and the oscillating part with the Bloch-Siegert shift with corrected frequency $\omega_0 - \frac{\Gamma^2}{2\omega_0}$.

8.4 A semi-classical treatment of Electromagnetically Induced Transparency (EIT) (*)[1]

1. The Hamiltonian is given by

$$H = \hbar\omega_{21}|2\rangle\,\langle 2\,| + \hbar\omega_{31}|\,3\rangle\,\langle 3| + e\mathbf{r}\cdot\mathbf{E} \tag{21}$$

So

$$\begin{aligned}
H|1\rangle &= e\mathbf{r}\cdot\mathbf{E}|1\rangle = \sum_j |j\rangle\langle j|e\mathbf{r}\cdot\mathbf{E}|1\rangle = \sum_j \mathbf{d}_{j1}\cdot\mathbf{E}|j\rangle \\
H|2\rangle &= \hbar\omega_{21}|2\rangle + e\mathbf{r}\cdot\mathbf{E}|2\rangle = \hbar\omega_{21}|2\rangle + \sum_j \mathbf{d}_{j2}\cdot\mathbf{E}|j\rangle \\
H|3\rangle &= \hbar\omega_{31}|3\rangle + e\mathbf{r}\cdot\mathbf{E}|3\rangle = \hbar\omega_{31}|3\rangle + \sum_j \mathbf{d}_{j3}\cdot\mathbf{E}|j\rangle
\end{aligned} \tag{22}$$

where $\mathbf{d}_{ij} = \langle i|e\mathbf{r}|j\rangle$ are the dipole matrices. Note that $\mathbf{d}_{ii} = 0$. We also consider decay rate terms in a_2 and a_3. Thus for a state $|\psi(t)\rangle = \sum_j a_j(t)|j\rangle, j = 1, 2, 3$, we have

$$\begin{aligned}
\dot{a}_1 &= -ia_2\frac{\mathbf{d}_{21}\cdot\mathbf{E}}{\hbar} - ia_3\frac{\mathbf{d}_{31}\cdot\mathbf{E}}{\hbar} \\
\dot{a}_2 &= -i\omega_{21}a_2 - \frac{\Gamma_2}{2}a_2 - ia_1\frac{\mathbf{d}_{12}\cdot\mathbf{E}}{\hbar} - ia_3\frac{\mathbf{d}_{32}\cdot\mathbf{E}}{\hbar} \\
\dot{a}_3 &= -i\omega_{31}a_3 - \frac{\Gamma_3}{2}a_3 - ia_1\frac{\mathbf{d}_{13}\cdot\mathbf{E}}{\hbar} - ia_2\frac{\mathbf{d}_{23}\cdot\mathbf{E}}{\hbar}
\end{aligned} \tag{23}$$

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[1] Graded exercise

2. We assume that $\omega_p - \omega_c \approx \omega_{21}$ and $\omega_p \approx \omega_{31}$. Take the following transformation

$$\begin{aligned} \tilde{a}_1 &= a_1 \\ a_2 &= \tilde{a}_2 e^{-i(\omega_p - \omega_c)t} \approx \tilde{a}_2 e^{-i\omega_{21}t} \\ a_3 &= \tilde{a}_3 e^{-i\omega_p t} \approx \tilde{a}_3 e^{-i\omega_{31}t} \end{aligned} \tag{24}$$

The electric field takes the form of $\mathbf{E} = 1/2\mathbf{E}_c e^{-i\omega_c t} + 1/2\mathbf{E}_p e^{-i\omega_p t} + cc.$. Then we have:

$$\begin{aligned} \dot{\tilde{a}}_1 &= -i\tilde{a}_3 \frac{\mathbf{d}_{31} \cdot \mathbf{E}_p}{2\hbar} \\ \dot{\tilde{a}}_2 &= (-i\omega_{21} + i\omega_p - i\omega_c - \frac{\Gamma_2}{2})\tilde{a}_2 - i\tilde{a}_3 \frac{\mathbf{d}_{32} \cdot \mathbf{E}_c}{2\hbar} \\ \dot{a}_3 &= (-i\omega_{31} + \omega_p - \frac{\Gamma_3}{2})\tilde{a}_3 - i\tilde{a}_1 \frac{\mathbf{d}_{13} \cdot \mathbf{E}_p}{2\hbar} - i\tilde{a}_2 \frac{\mathbf{d}_{23} \cdot \mathbf{E}_c}{2\hbar} \end{aligned} \tag{25}$$

3. Here we apply rotating wave approximation to cancel the fast rotating terms. Define $\Omega_{ij} = \mathbf{d}_{ij} \cdot \mathbf{E}/\hbar$ the Rabi frequency. The stationary equations are written as

$$\begin{aligned} \dot{a}_2 &= -i\tilde{\Delta}_2 a_2 - i\tilde{a}_3 \frac{\Omega_{32}}{2} = 0 \\ \dot{a}_3 &= -i\tilde{\Delta}_2 \tilde{a}_3 - i\tilde{a}_1 \frac{\Omega_{31}}{2} - i\tilde{a}_2 \frac{\Omega_{32}}{2} = 0 \end{aligned} \tag{26}$$

where we define $\tilde{\Delta}_2 = \omega_{21} - \omega_p + \omega_c - i\Gamma_2/2$ and $\tilde{\Delta}_3 = \omega_{31} - \omega_p - i\Gamma_3/2$. We assume that Ω_{21} and Ω_{31} are small compared to Ω_{23}. We also assume that $a_1 = a_0$. Such that we can solve a_2 and a_3 as:

$$\begin{aligned} \tilde{a}_3 &= \frac{2\Omega_{31}\tilde{\Delta}_2}{-4\tilde{\Delta}_2\tilde{\Delta}_3 + \Omega_{32}^2} a_0 \\ \tilde{a}_2 &= \frac{-\Omega_{31}\Omega_{32}}{-4\tilde{\Delta}_2\tilde{\Delta}_3 + \Omega_{32}^2} a_0 \end{aligned} \tag{27}$$

4. The dipole moment can be calculated through $a_1, \tilde{a}_2, \tilde{a}_3$ by

$$\begin{aligned} \mathbf{P} &= N\langle\psi|e\mathbf{r}|\psi\rangle = N\sum_{i\neq j} \tilde{a}_i^* \tilde{a}_j \mathbf{d}_{ij} \\ &= N\left(\mathbf{d}_{12} a_1^* a_2 e^{i\omega_{21}t} + \mathbf{d}_{13} a_1^* a_3 e^{i\omega_{31}t} + \mathbf{d}_{23} a_2^* a_3 e^{i(\omega_{31}-\omega_{21})t} + \text{ c.c}\right) \\ &= N\left(\mathbf{d}_{12} a_1^* a_2 e^{i\omega_{21}t} + \mathbf{d}_{13} a_1^* a_3 e^{i\omega_{31}t} + \mathbf{d}_{23} a_2^* a_3 e^{i\omega_{32}t} + \text{ c.c}\right) \end{aligned} \tag{28}$$

5. The ω_p component of dipole moment can be calculated by extracting the ω_p frequency components, which is

$$\begin{aligned} \chi(\omega_p) &= \frac{N|a_0|^2|\mathbf{d}_{13}|^2}{\epsilon_0\hbar} \frac{2\Omega_{31}\tilde{\Delta}_2}{-4\tilde{\Delta}_2\tilde{\Delta}_3 + \Omega_{32}^2} \\ &= \frac{N|a_0|^2|\mathbf{d}_{13}|^2}{\epsilon_0\hbar} \frac{2\Omega_{31}(\omega_{21} - \omega_p + \omega_c - i\Gamma_2/2)}{-4(\omega_{21} - \omega_p + \omega_c - i\Gamma_2/2)(\omega_{31} - \omega_p - i\Gamma_3) + \Omega_{32}^2} \end{aligned} \tag{29}$$

6. The group velocity is defined by

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{c}{n(\omega_p) + \omega_p \frac{\mathrm{d}n}{\mathrm{d}\omega_p}} \tag{30}$$

where $n\left(\omega\right.$ is the refractive index at frequency ω_p. The refractive index is related to the susceptibility by

$$n\left(\omega_p\right) = \sqrt{1 + \chi_R\left(\omega_p\right)} \approx 1 + \frac{\chi_R\left(\omega_p\right)}{2} \tag{31}$$

We assume that Ω_{23} is larger than any other appeared frequency. Since $\omega_p \approx \omega_{31}$ we have

$$v_g = \frac{c}{n\left(\omega_p\right) + \omega_p \frac{\mathrm{d}n}{\mathrm{d}\omega_p}} \approx \frac{c}{1 + \frac{N\left|\mathbf{d}_{31}^2\right|}{\epsilon_0 \hbar} \frac{2\omega_p}{\left|\Omega_{23}\right|^2}} = \frac{\hbar \epsilon_0 c \left|\Omega_{23}^2\right|}{N\left|\mathbf{d}_{31}\right|^2 2\omega_p + \epsilon_0 \hbar \left|\Omega_{23}^2\right|} \tag{32}$$

The frequency response of the refractive index is much smaller comparing to the pump frequency, therefore $\omega_p dn/d\omega_p \gg 1$. Hence

$$v_g \approx \frac{\hbar \epsilon_0 c \left|\Omega_{23}\right|^2}{N\left|\mathbf{d}_{31}\right|^2 \cdot 2\omega_p} \tag{33}$$