Quantum Electrodynamics and Quantum Optics

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

Exercise No.5

5.1 Balanced homodyne detection

5.1.1 Fluctuation of photon number differences

The difference of the photon number is given by

$$n_{21} = a_2''^{\dagger} a_2'' - a_1''^{\dagger} a_1'' \tag{1}$$

where

$$\begin{aligned} a_1''^{\dagger} &= \frac{1}{\sqrt{2}} \left(a'^{\dagger} + i a_{LO}^{\dagger} \right) \\ a_2''^{\dagger} &= \frac{1}{\sqrt{2}} \left(i a'^{\dagger} + a_{LO}^{\dagger} \right) \end{aligned} \tag{2}$$

Thus

$$n_{21} = a_2''^{\dagger} a_2'' - a_1''^{\dagger} a_1'' = i \left(a'^{\dagger} a_{LO} - a_{LO}^{\dagger} a' \right) \tag{3}$$

Assume that the local oscillator is a strong coherent state, i.e. $\alpha_{LO} = |\alpha_{LO}| e^{i\phi}$, we define the quadratures $X(\phi) = \frac{1}{2} \left(a'^{\dagger} e^{i\phi} + a' e^{-i\phi} \right)$. Thus we obtain

$$\begin{aligned} n_{21} &= i |\alpha_{LO}| \left(a'^{\dagger} e^{i\phi} - a' e^{-i\phi} \right) = 2 |\alpha_{LO}| X(\phi + \pi/2) \\ \langle n_{21}^2 \rangle &= 4 |\alpha_{LO}|^2 \langle X^2(\phi + \pi/2) \rangle \\ \langle n_{21} \rangle^2 &= 4 |\alpha_{LO}|^2 \langle X(\phi + \pi/2) \rangle^2 \end{aligned} \tag{4}$$

Therefore the fluctuation of n_{21} is

$$\begin{aligned} \Delta n_{21}^2 &= \langle n_{21}^2 \rangle - \langle n_{21} \rangle^2 = 4 |\alpha_{LO}|^2 \left(\langle X^2(\phi + \pi/2) \rangle - \langle X(\phi + \pi/2) \rangle^2 \right) \\ &= 4 |\alpha_{LO}|^2 \Delta X(\phi + \pi/2)^2 \end{aligned} \tag{5}$$

5.1.2 Classical amplitude noise

From (2) we know that

$$\begin{aligned} n_1 = a''^{\dagger} a'' &= \frac{1}{2} \left(a'^{\dagger} a' + a_{LO}^{\dagger} a_{LO} - i \left(a'^{\dagger} a_{LO} - a_{LO}^{\dagger} a' \right) \right) \\ n_2 = a''^{\dagger} a'' &= \frac{1}{2} \left(a'^{\dagger} a' + a_{LO}^{\dagger} a_{LO} + i \left(a'^{\dagger} a_{LO} - a_{LO}^{\dagger} a' \right) \right) \end{aligned} \tag{6}$$

where $n' = a'^{\dagger} a'$ and $n_{LO} = a_{LO}^{\dagger} a_{LO}$ are number of photons of signal and local oscillators. For a one-port normal homodyne detection, the output fluctuation consists of two parts: the interference term $\left(i \left(a'^{\dagger} a_{LO} - a_{LO}^{\dagger} a' \right) \right)$ and the non-interference terms $(n' + n_{LO})$. Thus if we add a Gaussian noise in the local oscillator δN, the output signal will contain a similar term. Note that, however, the non-interference terms $n' + n_{LO}$ cancel each other in $n_{21} = n_2 - n_1$, which means a noise in the local oscillator completely vanishes in the output detection.

5.1.3 Squeeze coherent state

Write a' in terms of a and b, ladder operators of signal and the vacuum state

$$\begin{aligned} a'^{\dagger} &= \sqrt{T} a^{\dagger} + i \sqrt{1 - T} b^{\dagger} \\ a' &= \sqrt{T} a - i \sqrt{1 - T} b \end{aligned} \tag{7}$$

Thus

$$X(\phi+\pi/2) = \sqrt{T}X_a(\phi+\pi/2) + i\sqrt{1-T}X_b(\phi+\pi/2) \tag{8}$$

Therefore we can derive

$$\Delta X(\phi+\pi/2)^2 = T\Delta X_a(\phi+\pi/2)^2 + (1-T)\Delta X_b(\phi+\pi/2)^2 \tag{9}$$

Since the signal is a squeezed state and the vacuum state is a coherent state, we have (along the squeezing angle)

$$\begin{aligned}\Delta X_a(\phi+\pi/2)^2 &= \frac{1}{2}e^{-2r}\\ \Delta X_b(\phi+\pi/2)^2 &= \frac{1}{2}e^{2r}\end{aligned} \tag{10}$$

Substitute this into (10), we obtain

$$\Delta X(\phi+\pi/2)^2 = \frac{T}{2}e^{-2r} + \frac{1-T}{2}e^{2r} = \frac{1}{2}\cosh(2r) + \frac{1-2T}{2}\sinh(2r) \tag{11}$$

and

$$\Delta n_{21}^2 = 2\left|\alpha_{LO}\right|^2\left(\cosh(2r) + (1-2T)\sinh(2r)\right) \tag{12}$$

5.2 Solution: Squeezing in homodyne detection

After the 50:50 beamsplitter, the annihilation operator associated to the signal becomes $\hat{c} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{b})$[1]. For simplicity (the proportionality factor will not matter in the SNR anyway), the photocurrent is defined as

$$\hat{I} = \hat{c}^\dagger\hat{c} = \frac{1}{2}\left(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + \hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}\right) \tag{13}$$

Since the state entering the beamsplitter is the separable state $|\psi\rangle_a|\beta\rangle_b$ and that the operators $\hat{a}$ and $\hat{b}$ commute (since they act on distinct modes), the total expectation values can be decomposed as a sum of products of expectation values over a and over b. Indeed the mean of the photocurrent is

$$\begin{aligned}\langle\hat{I}\rangle &= \frac{1}{2}\left(\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{b}\rangle + \langle\hat{a}^\dagger\hat{b}\rangle + \langle\hat{b}^\dagger\hat{a}\rangle\right)\\ &= \frac{1}{2}\left(\langle\hat{a}^\dagger\hat{a}\rangle + |\beta|^2 + \beta\langle\hat{a}^\dagger\rangle + \beta^*\langle\hat{a}\rangle\right)\end{aligned} \tag{14}$$

Similarly, for the variance we first compute $\langle\hat{I}^2\rangle$ and $\langle\hat{I}\rangle^2$ since $\Delta\hat{I}^2 = \langle(\hat{I}-\langle\hat{I}\rangle)^2\rangle = \langle\hat{I}^2\rangle - \langle\hat{I}\rangle^2$

$$\begin{aligned}4\langle\hat{I}^2\rangle =& \langle\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{b}\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{b}\rangle + \langle\hat{b}^\dagger\hat{b}\hat{a}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{a}\hat{b}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{a}\hat{a}^\dagger\hat{b}\rangle\\ &+ \langle\hat{a}^\dagger\hat{b}\hat{b}^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{b}\hat{a}^\dagger\hat{b}\rangle + \langle\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\rangle + \langle\hat{b}^\dagger\hat{b}\hat{b}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{b}\hat{a}^\dagger\hat{b}\rangle + \langle\hat{b}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\rangle + \langle\hat{a}^\dagger\hat{b}\hat{b}^\dagger\hat{b}\rangle + \langle\hat{b}^\dagger\hat{b}\hat{b}^\dagger\hat{b}\rangle\\ =& \langle\left(\hat{a}^\dagger\right)^2\hat{a}^2\rangle + 2\langle\hat{a}^\dagger\hat{a}\rangle + 2\beta^*\left(\langle\hat{a}^\dagger\hat{a}^2\rangle + \langle\hat{a}\rangle\right) + 2\beta\left(\langle\left(\hat{a}^\dagger\right)^2\hat{a}\rangle + \langle\hat{a}^\dagger\rangle\right) + 2|\beta|^2\left(2\langle\hat{a}^\dagger\hat{a}\rangle + 1\right)\\ &+ (\beta^*)^2\langle\hat{a}^2\rangle + \beta^2\langle\left(\hat{a}^\dagger\right)^2\rangle + 2\beta^*|\beta|^2\langle\hat{a}\rangle + 2\beta|\beta|^2\langle\hat{a}^\dagger\rangle + |\beta|^4\\ 4\langle\hat{I}\rangle^2 =& \langle\hat{a}^\dagger\hat{a}\rangle^2 + 2\beta^*\langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{a}\rangle + 2\beta\langle\hat{a}^\dagger\rangle\langle\hat{a}^\dagger\hat{a}\rangle + 2|\beta|^2\langle\hat{a}^\dagger\hat{a}\rangle + 2|\beta|^2\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle + (\beta^*)^2\langle\hat{a}\rangle^2 + \beta^2\langle\hat{a}^\dagger\rangle^2\\ &+ 2\beta^*|\beta|^2\langle\hat{a}\rangle + 2\beta|\beta|^2\langle\hat{a}^\dagger\rangle + |\beta|^4\end{aligned} \tag{15}$$

[1]Note that the scattering matrix of a 50:50 beamsplitter is not uniquely defined. Here we use one of the possible forms

Then

$$
\begin{aligned}
4\Delta\hat{I}^2 &= \langle\hat{I}^2\rangle - \langle\hat{I}\rangle^2 \\
&= \Delta\hat{n}^2 + \langle\hat{a}^\dagger\hat{a}\rangle + 2\beta^*\left(\langle\hat{a}^\dagger\hat{a}^2\rangle - \langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{a}\rangle + \langle\hat{a}\rangle\right) + 2\beta\left(\langle\left(\hat{a}^\dagger\right)^2\hat{a}\rangle - \langle\hat{a}^\dagger\rangle\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\rangle\right) \\
&+ 2|\beta|^2\left(\langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle + 1\right) + (\beta^*)^2\left(\langle\hat{a}^2\rangle - \langle\hat{a}\rangle^2\right) + \beta^2\left(\langle\left(\hat{a}^\dagger\right)^2\rangle - \langle\hat{a}^\dagger\rangle^2\right)
\end{aligned}
\tag{16}
$$

(a) Consider the input signal is a coherent state $|\alpha\rangle$: $\Delta\hat{n}^2 = |\alpha|^2$, then

$$
\begin{aligned}
\langle\hat{I}\rangle &= \frac{1}{2}\left(|\alpha|^2 + |\beta|^2 + \beta\alpha^* + \beta^*\alpha\right) = \frac{1}{2}|\alpha+\beta|^2 \\
4\Delta\hat{I}^2 &= 2\left(|\alpha|^2 + \beta^*\alpha + \beta\alpha^* + |\beta|^2\right) = 2|\alpha+\beta|^2
\end{aligned}
\tag{17}
$$

This yields an SNR of

$$
\mathrm{SNR} = \frac{\langle\hat{I}\rangle}{\sqrt{\Delta\hat{I}^2}} = \frac{1}{\sqrt{2}}|\alpha+\beta| \tag{18}
$$

and when $|\beta| \gg |\alpha|$, this reduces to $\mathrm{SNR} = \frac{1}{\sqrt{2}}|\beta|$.

(b) For a squeezed vacuum state $|0,\xi\rangle = \hat{S}(\xi)|0\rangle$ with $\xi = re^{i\theta}$, we have

$$
\begin{aligned}
&\langle\hat{a}\rangle = \langle\hat{a}^\dagger\rangle = \langle\hat{a}^2\rangle = \langle\left(\hat{a}^\dagger\right)^2\rangle = \langle\hat{a}^\dagger\hat{a}^2\rangle = \langle\left(\hat{a}^\dagger\right)^2\hat{a}\rangle = 0 \\
&\langle\hat{n}\rangle = \sinh^2(r) \\
&\Delta\hat{n}^2 = 2\cosh^2(r)\sinh^2(r)
\end{aligned}
\tag{19}
$$

Therefore, we compute

$$
\begin{aligned}
\langle\hat{I}\rangle &= \frac{1}{2}\left(\sinh^2(r) + |\beta|^2\right) \\
4\Delta\hat{I}^2 &= 2\cosh^2(r)\sinh^2(r) + \sinh^2(r) + 2|\beta|^2\left(\sinh^2(r) + 1\right) \\
&= 2\cosh^2(r)\left(\sinh^2(r) + |\beta|^2\right) + \sinh^2(r)
\end{aligned}
\tag{20}
$$

Which gives the following SNR :

$$
\mathrm{SNR} = \frac{\langle\hat{I}\rangle}{\sqrt{\Delta\hat{I}^2}} = \frac{\sinh^2(r) + |\beta|^2}{\sqrt{2\cosh^2(r)\left(\sinh^2(r) + |\beta|^2\right) + \sinh^2(r)}} \tag{21}
$$

5.3 Wigner function marginal distributions

1. The integral of Wigner function in momentum space is given by

$$
\begin{aligned}
\int_{-\infty}^{\infty} dp W(x,p) &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} du e^{-ipu/\hbar}\psi^*\left(x - \frac{u}{2}\right)\psi\left(x + \frac{u}{2}\right) \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} du \left(\int_{-\infty}^{\infty} dp e^{-ipu/\hbar}\right)\psi^*\left(x - \frac{u}{2}\right)\psi\left(x + \frac{u}{2}\right) \\
&= \int_{-\infty}^{\infty} du \delta(u)\psi^*\left(x - \frac{u}{2}\right)\psi\left(x + \frac{u}{2}\right) \\
&= |\psi(x)|^2
\end{aligned}
\tag{22}
$$

where from the second line to the third line we use the definition of Dirac delta function. Same procedure can be applied to calculate the marginal distribution on position space by rewriting the wave function in the momentum representation.

2. The Wigner fucntion of a plane wave $\psi(x) = (2\pi\hbar)^{-1}\exp(-i\mathbf{p}x/\hbar)$ is given by

$$W(x,p) = \int \frac{du}{(2\pi\hbar)^2} e^{i\mathbf{p}(x+u/2)/\hbar} e^{-i\mathbf{p}(x-u/2)/\hbar} e^{-ipu/\hbar} = \int \frac{du}{(2\pi\hbar)^2} e^{-i(p-\mathbf{p})u/\hbar} = \frac{1}{2\pi\hbar}\delta(p-\mathbf{p}) \tag{23}$$

The Wigner fucntion of a stationary wave $\psi(x) = (\pi\hbar)^{-1/2}\cos(\mathbf{p}x/\hbar)$ is given by

$$\begin{aligned} W(x,p) &= \int \frac{du}{2(\pi\hbar)^2} \cos\left(\frac{\mathbf{p}(x+u/2)}{\hbar}\right) \cos\left(\frac{\mathbf{p}(x-u/2)}{\hbar}\right) e^{-ipu/\hbar} \\ &= \int \frac{du}{(2\pi\hbar)^2} \left(\cos(\mathbf{p}u/\hbar) + \cos(2\mathbf{p}x/\hbar)\right) e^{-ipu/\hbar} \\ &= \int \frac{du}{(2\pi\hbar)^2} \left(e^{-i\mathbf{p}u/\hbar} + e^{i\mathbf{p}u/\hbar} + \cos(2\mathbf{p}x/\hbar)\right) e^{-ipu/\hbar} \\ &= \frac{1}{2\pi\hbar} \left(\delta(p-\mathbf{p}) + \delta(p+\mathbf{p}) + \cos(2\mathbf{p}x/\hbar)\delta(p)\right) \end{aligned} \tag{24}$$