Quantum Electrodynamics and Quantum Optics
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

Exercise No.5

5.1 Balanced Homodyne Detection

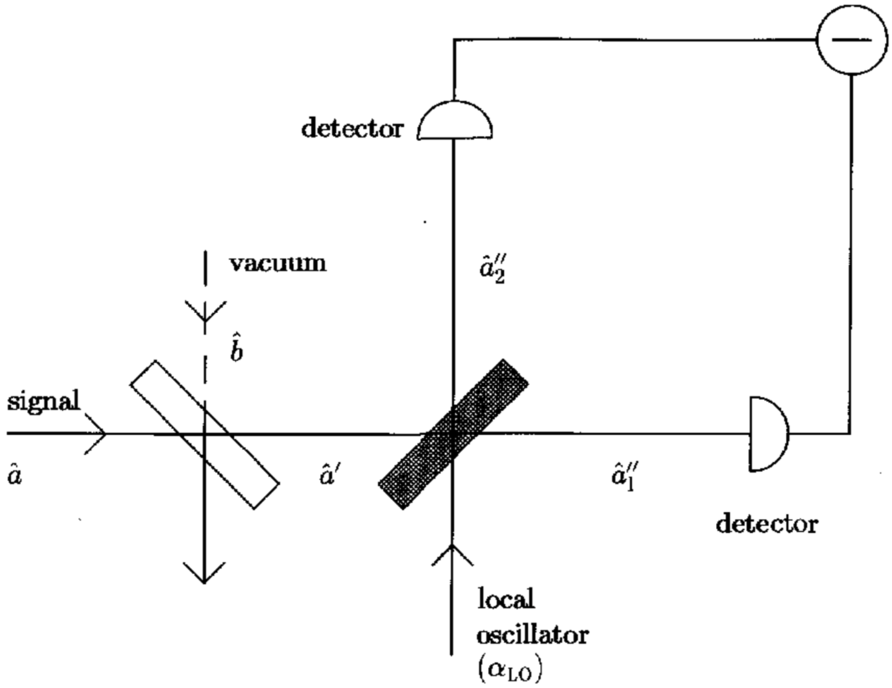

Figure 1: Scheme of a balanced homodyne detection.

The balanced homodyne detection[1] is simply the balanced version of homodyne detection. It has the great practical advantage of canceling technical amplitude noise in the local oscillator. The principle scheme of the balanced homodyne detector is depicted in Fig. 1. The signal interferes with a coherent laser beam (local oscillator) $|\alpha_{\text{LO}}\rangle$ with much larger intensity at a well-balanced $50:50$ beam splitter. After the optical mixing of the signal with the local oscillator, each emerging beam is directed to a photodetector. The photocurrents I_1 and I_2 are measured, electronically processed and finally subtracted from each other. The difference $I_{21} = I_2 - I_1$ is the quantity of interest because it contains the interference term of the local oscillator and the signal. We assume for the ideal case that the measured photocurrents I_1 and I_2 are proportional to the photon numbers $\hat{n}_1$ and $\hat{n}_2$ of the beams incident on each detector.

1. First by neglecting the added beam splitter at the signal path, and assuming perfect detection efficiencies of both detectors, try to calculate the fluctuation of the photon number difference $\langle \Delta \hat{n}_{21}^2 \rangle = \langle \Delta(\hat{n}_2 - \hat{n}_1)^2 \rangle = \langle \Delta(|\alpha_{\text{LO}}|\hat{X}^{\text{sig}}_{\theta_{\text{LO}}})^2 \rangle$ which is proportional to the detected photocurrent I_{21}. Here we consider a perfect $50:50$ ratio beam splitter and a much stronger coherent local oscillator.

2. Now show that at the presence of classical local oscillator amplitude noise (Gaussian) of the mean photon flux $\langle N_{\text{LO}} \rangle = \langle n_{\text{LO}} \rangle + \delta N$, the balanced scheme suppresses the classical noise compared to a normal homodyne scheme with just a single detector.

3. In real experiment, the detection efficiency are not perfect, and signal loss can affect the detection noise feature. Here we model the signal loss by inserting a beam splitter with Fresnel

[1] The exercise was made based on "Measuring the quantum state of light" by Ulf Leonhardt Chap 4.

coefficients r and t. The other input port of the beam splitter is vacuum as illustrated in Fig. 1. Calculate and show that if the signal is a squeezed coherent state $|\alpha, \xi\rangle$, the squeezing effect detected by the balanced homodyne at different optical quadratures will degrade due to the vacuum input through the added beam splitter.

5.2 Squeezing in homodyne detection

Consider a homodyne detection setup as shown in figure 2. The signal is injected into the beam-splitter through port a (with annihilation operator $\hat{a}$), and local oscillator is injected through port b (with annihilation operator $\hat{b}$). The output light at port c (with annihilation operator $\hat{c}$) is detected with a photodetector. For simplicity assume the beamsplitter to be 50:50 (η = 0.5). The local oscil-

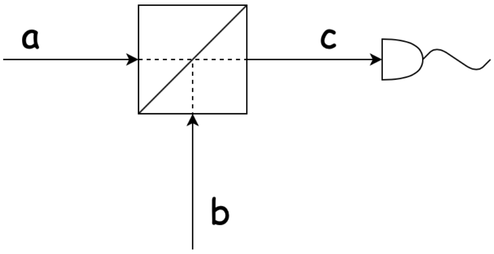

Figure 2: a homodyne detection setup

lator is a strong coherent state $|\beta\rangle$. The output photocurrent of the photodetector, I is proportional to $\hat{c}^\dagger\hat{c}$ (for simplicity assume $\hat{I} = \hat{c}^\dagger\hat{c}$).

(a) Consider the input signal is a coherent state $|\alpha\rangle$. Compute mean and variance for the photocurrent, $\langle\hat{I}\rangle$ and $\Delta I^2 = \langle(\hat{I} - \langle\hat{I}\rangle)^2\rangle$. Compute the signal to noise ratio (SNR) defined as $SNR = \frac{\langle\hat{I}\rangle}{\sqrt{\Delta I^2}}$ and simplify it for $|\beta| \gg |\alpha|$.

(b) Repeat the previous part for a squeezed input state $|0, \xi\rangle = \hat{S}(\xi)\,|0\rangle$.

5.3 Wigner function marginal distributions

1. An arbitary state $|\alpha\rangle$ can be projected on coordinate and momentum states $|x\rangle$ and $|p\rangle$ to obtain coordinate- and momentum-state wavefunctions $\psi(x) = \langle x|\alpha\rangle$ and $\psi(p) = \langle p|\alpha\rangle$. These wafefunctions are related to each other through Fourier transform:

$$\psi(p) = \int \frac{dx}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\psi(x). \tag{1}$$

$|\psi(x)|^2$ and $|\psi(p)|^2$ represent the probability distribution functions for position and momentum. Can these distributions be obtained as marginal distributions from a joint probability for position and momentum? No, since the existence of such a function would imply that position and momentum may be simultaneously well defined! However, the **Wigner function** is very close to such an entity for a pure state $\rho = |\psi\rangle\langle\psi|$:

$$W(x,p) = \int \frac{du}{2\pi\hbar} e^{-ipu/\hbar}\psi^*\left(x - \frac{u}{2}\right)\psi\left(x + \frac{u}{2}\right), \tag{2}$$

since it indeed gives the marginals upon integration:

$$\int dp\, W(x,p) = |\psi(x)|^2, \tag{3a}$$

$$\int dx\, W(x,p) = |\psi(p)|^2. \tag{3b}$$

Show the results from Eqs. (3).

The dynamics of $W(x,p)$ is quite remarkable, since it is very similar to classical transport equation:

$$\frac{\partial W}{\partial t} = -\{H, W\} + O(\hbar^2), \tag{4}$$

where $\{H, W\}$ is a Poisson bracket. However, one should still keep in mind that W cannot be considered a classical function, since it may be negative in some regions of phase space.[2]

2. Now let us consider some particular examples.

   (a) For a **momentum state**

   $$\psi(x) = \frac{e^{-i\mathbf{p}x/\hbar}}{\sqrt{2\pi\hbar}}, \tag{5}$$

   one can obtain the following Wigner function:

   $$W(q,p) = \frac{1}{2\pi\hbar}\delta(p - \mathbf{p}), \tag{6}$$

   (b) For a **stationary wave**, which is a coherent superposition of two states of opposite momentum,

   $$\psi(x) = \frac{1}{\sqrt{\pi\hbar}} \cos \mathbf{p}x/\hbar, \tag{7}$$

   one can obtain:

   $$W(q,p) = \frac{1}{4\pi\hbar}\left[\delta(p - \mathbf{p}) + \delta(p + \mathbf{p})\right] + \cos\left(\frac{2\mathbf{p}x}{\hbar}\right)\delta(p). \tag{8}$$

   Show the results from Eqs. (6) and (8) and try to explain why these two cases are different.

5.4 Wigner-Ville distribution(*)[3]

The Wigner distribution was introduced to signal analysis by Ville[4] inspired by the Wigner paper. In this area of science, it is called Wigner-Ville distribution, defined in terms of signal, $s(t)$, and its spectrum, $S(\omega)$:[5]

$$W(t,\omega) = \frac{1}{2\pi}\int d\tau s^*(t-\tau/2)s(t+\tau/2)e^{-i\omega\tau} \tag{9a}$$

$$= \frac{1}{2\pi}\int d\theta S^*(\omega+\theta/2)S(\omega-\theta/2)e^{-i\theta t}. \tag{9b}$$

1. Knowing that signal and its spectrum are related to each other via Fourier transform:

   $$s(t) = \frac{1}{\sqrt{2\pi}}\int S(\omega)e^{i\omega t}d\omega, \tag{10}$$

   prove the equivalence of Eqs. (9a) and (9b).

2. Show that the **Wigner distribution is real**, even if the signal is complex.

3. For which condition on $s(t)$, the Wigner-Ville distribution will be equal for frequencies ω and $-\omega$? And, on the other side, which condition must be satisfied for $S(\omega)$ to have Wigner-Ville distribution symmetric in time t? *Hint: real signals have symmetrical spectra and vice versa.*

[2] For more information, see Nonequilibrium Quantum Field Theory book by Calzetta and Bei-Lok (part 3.3.1 Wigner functions)

[3] Graded exercise

[4] Ville, J. "Theorie et application dela notion de signal analytique." Cables et transmissions 2.1 (1948): 61-74.

[5] For more information, see Time-Frequency Analysis book by Cohen (Chapter 8 The Wigner distribution)

4. Show that the Wigner-Ville distribution satisfies time-frequency marginals:

$$\int W(t,\omega)d\omega = |s(t)|^2, \tag{11a}$$

$$\int W(t,\omega)dt = |S(\omega)|^2. \tag{11b}$$

Hint: do it in the same manner as for the quantum-mechanical Wigner function relatively to coordinate- and momentum-domain probabilities.

5. Show that the first mixed moment of the Wigner-Ville function,

$$\langle t\omega \rangle = \iint t\omega \, W(t,\omega) \, dtd\omega, \tag{12}$$

gives the covariance of the signal $\mathrm{Cov}_{t\omega} \propto \langle t\omega_i \rangle$, where ω_i is an instantaneous frequency. *Hint: integrate over one of the variables.*

6. Try to plot the Wigner-Ville distribution of the following example signal:

$$\begin{array}{cc} 0 < t < t_1 & A\cos\omega_1 t \\ t_1 < t < t_2 & 0 \\ t_2 < t < t_3 & A\cos\omega_2 t \\ \text{with } A = 1 & \omega_i = 2\pi \times 10i \end{array}$$