Quantum Electrodynamics and Quantum Optics
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Solutions to Exercise No.1

1.1 Solution: Classical electromagnetic field modes density in free space and field quantization

Solution: Classical electromagnetic field modes density in free space and field quantization (1) Wave equation for the vector potential In vacuum (SI units) Maxwell's equations read

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = 0, \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0. \tag{2}$$

Introduce the potentials $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$. In the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ and with zero charge density we have $\nabla^2\phi = 0$. With suitable boundary conditions we set $\phi = 0$, hence $\mathbf{E} = -\dot{\mathbf{A}}$. Taking the curl of Faraday's law $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$ gives

$$\nabla \times (\nabla \times \mathbf{A}) = -\frac{\partial(\nabla \times \mathbf{A})}{\partial t}. \tag{3}$$

Using the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$ and the Coulomb gauge, we obtain

$$-\nabla^2\mathbf{A} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}. \tag{4}$$

Therefore the (three-component) vector potential satisfies the wave equation

$$\nabla^2\mathbf{A}(\mathbf{r}, t) - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} = 0, \qquad c = \frac{1}{\sqrt{\mu_0\epsilon_0}}. \tag{5}$$

Because any solution can be decomposed into two transverse polarizations, each Cartesian component along a fixed polarization obeys the scalar wave equation

$$\nabla^2 A(\mathbf{r}, t) - \frac{1}{c^2}\frac{\partial^2 A(\mathbf{r}, t)}{\partial t^2} = 0. \tag{6}$$

(2) Solutions in a cube of edge L Solve Eq. 6 in a cube using separation of variables and impose periodic boundary conditions (PBC) (equivalently take a metallic box and the result is identical in the large-L limit). With PBC the normal modes are plane waves

$$A(\mathbf{r}, t) = \mathcal{A}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} + \text{c.c.}, \qquad \omega = c\,\|\mathbf{k}\|, \tag{7}$$

and periodicity requires

$$k_x = \frac{2\pi n_x}{L}, \qquad k_y = \frac{2\pi n_y}{L}, \qquad k_z = \frac{2\pi n_z}{L}, \qquad n_i \in \mathbb{Z}. \tag{8}$$

Each admissible $\mathbf{k} \neq 0$ supports *two* independent transverse polarizations (since $\mathbf{k} \cdot \mathbf{A} = 0$).

(3) Counting modes in free space Let the box become very large ($L \to \infty$, fixed $V = L^3$) and count the number of distinct $\mathbf{k}$ with magnitude in $[k, k + dk]$. In $\mathbf{k}$-space the allowed points form a cubic lattice with volume per point $\left(\frac{2\pi}{L}\right)^3 = \frac{(2\pi)^3}{V}$. Hence the number of lattice points in a spherical shell of volume $4\pi k^2\, dk$ is

$$\frac{V}{(2\pi)^3}\, 4\pi k^2\, dk. \tag{9}$$

Including two polarizations gives

$$dN = 2 \times \frac{V}{(2\pi)^3} 4\pi k^2\, dk = \frac{V}{\pi^2} k^2\, dk. \tag{10}$$

(4) Mode density $\rho(\nu)$ Relate k to the ordinary frequency ν via $\omega = ck = 2\pi\nu$ so that

$$k = \frac{2\pi\nu}{c}, \qquad dk = \frac{2\pi}{c}\, d\nu. \tag{11}$$

Divide Eq. 10 by $V\, d\nu$ to obtain the 3D mode density per unit frequency per unit volume:

$$\rho(\nu) \equiv \frac{dN}{V\, d\nu} = \frac{k^2}{\pi^2}\frac{dk}{d\nu} = \frac{1}{\pi^2}\left(\frac{2\pi\nu}{c}\right)^2\left(\frac{2\pi}{c}\right) = \boxed{\frac{8\pi\,\nu^2}{c^3}}. \tag{12}$$

Therefore the characteristic dependence is $\rho(\nu) \propto \nu^2$.

(5) Rayleigh–Jeans law from equipartition Each normal mode of the EM field is a harmonic oscillator. In classical thermal equilibrium, the equipartition theorem assigns average energy $k_\mathrm{B}T$ per mode ($k_\mathrm{B}T/2$ in the "kinetic" electric part plus $k_\mathrm{B}T/2$ in the "potential" magnetic part). Thus the spectral energy density (energy per volume per frequency) is

$$u(\nu)\, d\nu = \rho(\nu)\, k_\mathrm{B}T\, d\nu = \boxed{\frac{8\pi\,\nu^2}{c^3} k_\mathrm{B}T\, d\nu}. \tag{13}$$

This is the Rayleigh–Jeans law. In wavelength form ($\nu = c/\lambda$) it reads

$$u(\lambda)\, d\lambda = \frac{8\pi k_\mathrm{B}T}{\lambda^4}\, d\lambda. \tag{14}$$

Classically this diverges at high frequency (ultraviolet catastrophe); in quantum theory each mode's energy is quantized, $\hat{H} = \sum_{\mathbf{k},\lambda} \hbar\omega_\mathbf{k} \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$, leading to Planck's law upon taking a thermal average.

e) The Hamiltonian of a 1D harmonics oscillator can be expressed as

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega x^2$$

According to equipartition theorem, every quadratic term of p or x contributes $1/2k_BT$ to the energy, so the total energy should be $\epsilon = k_BT$. Hence, we can give Rayleigh-Jeans law

$$\frac{\mathrm{d}E}{V\,\mathrm{d}\nu} = \epsilon\rho(\nu) = \frac{8\pi\nu^2 k_BT}{c^3}$$

f) By replacing $u(\boldsymbol{r}) = \alpha(\boldsymbol{r}, t) = \alpha e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{r}}$, we can rewrite the vector potential can be expressed as a superposition of α_k and α_k^*

$$\boldsymbol{A} = \sum_{k,\lambda} C\left[\hat{\epsilon}_k\alpha_k e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{r}} + \hat{\epsilon}_k^*\alpha_k^* e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{r}}\right]$$

where C is a constant and $\hat{\epsilon}$ is a direction vector and $\lambda = \pm 1$ indicates the polarization. Due to relation (2) we obatin

$$\boldsymbol{E} = \sum_{k,\lambda} i\omega C\left[\hat{\epsilon}_k\alpha_k e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{\epsilon}_k^*\alpha_k^* e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{r}}\right]$$

$$\boldsymbol{B} = \sum_{k,\lambda} iC\left[(\boldsymbol{k}\times\hat{\epsilon}_k)\,\alpha_k e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{r}} - (\boldsymbol{k}\times\hat{\epsilon}_k^*)\,\alpha_k^* e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{r}}\right]$$

Here we use $\boldsymbol{\nabla} \times \left(\hat{\epsilon}_k e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right) = i\,(\boldsymbol{k}\times\hat{\epsilon}_k)\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$. So

$$
\begin{aligned}
|\boldsymbol{E}|^2 =& \boldsymbol{E}\cdot\boldsymbol{E}^* \\
=& \sum_{k,\lambda} i\omega C\left[\hat{\epsilon}_k\alpha_k e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{\epsilon}_k^*\alpha_k^* e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right] \\
& \sum_{k,\lambda}\cdot(-i\omega)C^*\left[\hat{\epsilon}_k\alpha_k^* e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{\epsilon}_k^*\alpha_k e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}}\right] \\
=& \sum_{k,\lambda}\omega^2|C|^2\left(\alpha_k\alpha_k^* + \alpha_k^*\alpha_k\right)
\end{aligned}
$$

and

$$
\begin{aligned}
|\boldsymbol{B}|^2 =& \boldsymbol{B}\cdot\boldsymbol{B}^* \\
=& i\sum_{k,\lambda} C\left[\left(\boldsymbol{k}\times\hat{\epsilon}_k\right)\alpha_k e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - \left(\boldsymbol{k}\times\hat{\epsilon}_k^*\right)\alpha_k^* e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right] \\
& \cdot(-i)\sum_{k} C^*\left[\left(\boldsymbol{k}\times\hat{\epsilon}_k\right)\alpha_k^* e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}} - \left(\boldsymbol{k}\times\hat{\epsilon}_k^*\right)\alpha_k e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}}\right] \\
=& \sum_{k,\lambda}|\boldsymbol{k}|^2|C|^2\left(\alpha_k\alpha_k^* + \alpha_k^*\alpha_k\right)
\end{aligned}
$$

Then the Hamiltonian writes by using (67) and (23)
$H = \frac{1}{2}\int_V\left(\epsilon_0|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2\right)\mathrm{d}V = \frac{1}{2}\epsilon_0\int_V\left(|\boldsymbol{E}|^2 + c^2|\boldsymbol{B}|^2\right)\mathrm{d}V = \sum_{k,\lambda}|C|^2 V\epsilon_0\omega^2\left(\alpha\alpha^* + \alpha^*\alpha\right)$

In order to make the Hamiltonian have the unit of energy, we take $C = \sqrt{\frac{\hbar}{2\omega V\epsilon_0}}$. So, finally we obtain

$$
H = \sum_k\sum_{\lambda=\pm1}\hbar\omega\left(\alpha_k\alpha_k^* + \alpha_k^*\alpha_k\right)
$$

g) In order to follow the symmetric postulate we keep the form of $\alpha\alpha* + \alpha^*\alpha$. By replacing α and α^* by operators a and $a^\dagger$, the Hamiltonian becomes

$$
H = \sum_k\sum_{\lambda=\pm1}\hbar\omega\left(a_k a_k^\dagger + a_k^\dagger a_k\right)
$$

For electric field operator and magnetic field operator

$$
\boldsymbol{E} = \sum_{k,\lambda} i\omega\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}\left[\hat{\epsilon}_k a_k e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{\epsilon}_k^* a_k^\dagger e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]
$$

$$
\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} = \sum_{k,\lambda}\frac{i}{\mu_0}\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}\left[\left(\boldsymbol{k}\times\hat{\epsilon}_k\right)a_k e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - \left(\boldsymbol{k}\times\hat{\epsilon}_k^*\right)a_k^\dagger e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]
$$

h) Due to (27) and (28) and the facts that

$$
\begin{aligned}
\left[a_k, a_l\right] &= \left[a_k^\dagger, a_l^\dagger\right] = 0 \\
\left[a_k, a_l^\dagger\right] &= \delta_{k,l}
\end{aligned}
$$

the commutation relationship of different components of $\boldsymbol{E}$ and $\boldsymbol{B}$, i.e $\left[\boldsymbol{E}_{ki}, \boldsymbol{B}_{lj}\right]$ should be

$$
\begin{aligned}
\left[\boldsymbol{E}_{ki}(\boldsymbol{r}), \boldsymbol{B}_{lj}\left(\boldsymbol{r}'\right)\right] =& -\frac{\hbar}{2V\epsilon_0}\sum_{k,l,\lambda}\left[\epsilon_k^{(\lambda)}\otimes\left(\boldsymbol{l}\times\epsilon_l^{(\lambda)}\right)\left[a_k, a_l^\dagger\right] - \epsilon_k^{(\lambda)}\otimes\left(\boldsymbol{l}\times\epsilon_l^{(\lambda)}\right)\left[a_k^\dagger, a_l\right]\right] \\
& \times\left[e^{i(\boldsymbol{k}-\boldsymbol{l})\cdot(\boldsymbol{r}-\boldsymbol{r}')} - e^{-i(\boldsymbol{k}-\boldsymbol{l})\cdot(\boldsymbol{r}-\boldsymbol{r}'}\right)\right] \\
=& -\frac{\hbar}{2V\epsilon_0}k_z\left[e^{i(\boldsymbol{k}-\boldsymbol{l})\cdot(\boldsymbol{r}-\boldsymbol{r}')} - e^{-i(\boldsymbol{k}-\boldsymbol{l})\cdot(\boldsymbol{r}-\boldsymbol{r}'}\right)\right]
\end{aligned}
$$

Replace the sum of k by integral we will have

$$\left[\boldsymbol{E}_{ki}(\boldsymbol{r}), \boldsymbol{B}_{lj}\left(\boldsymbol{r}'\right)\right] = \delta_{k,l}\delta_{r,r} = -i\frac{\hbar}{\epsilon_0}\delta_{k,k'}\frac{\partial}{\partial k}\delta_{r,r'}\epsilon_{ijk}$$

i) For vacuum state

$$\langle E\rangle = \langle 0|\boldsymbol{E}|0\rangle = \left\langle 0\left| i\omega\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}\left[\hat{\epsilon}_k a_k e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{\epsilon}_k^* a_k^\dagger e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]\right| 0\right\rangle = 0$$

Since $|\boldsymbol{E}|^2 = \sum_k \frac{\hbar\omega}{2V\epsilon_0}(n_k + 1/2)$, so

$$\left\langle (E - \langle E\rangle)^2\right\rangle = \left\langle 0\left|(E-\langle E\rangle)^2\right|0\right\rangle = \left\langle n_k\left|E^2\right|n_k\right\rangle = \frac{\hbar\omega}{4V\epsilon_0}$$

For a higher state $|n_k\rangle$,

$$\langle E\rangle = \langle n_k|\boldsymbol{E}|n_k\rangle = \left\langle n_k\left| i\omega\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}\left[\hat{\epsilon}_k a_k e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{\epsilon}_k^* a_k^\dagger e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]\right| n_k\right\rangle = 0$$

and

$$\left\langle (E - \langle E\rangle)^2\right\rangle = \left\langle n_k\left|(E-\langle E\rangle)^2\right|n_k\right\rangle = \left\langle n_k\left|E^2\right|n_k\right\rangle = \frac{\hbar\omega}{2V\epsilon_0}(n_k + 1/2)$$

1.2 Solution: Review on commutation relations and operators

The first two commutators are proven by induction and the last three build upon the previously derived commutation relations.

1. Let us demonstrate that $\left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] = n\left(\hat{a}^\dagger\right)^{n-1}$ by induction,
   For $n = 0:$ $[\hat{a}, 1] = 0$
   For $n = 1:$ $\left[\hat{a}, \hat{a}^\dagger\right] = 1$
   For $n = 2:$ $\left[\hat{a}, \left(\hat{a}^\dagger\right)^2\right] = \hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] + [\hat{a}, \hat{a}^\dagger]\hat{a}^\dagger = 2\hat{a}^\dagger$
   For $n = 3:$ $\left[\hat{a}, \left(\hat{a}^\dagger\right)^3\right] = \ldots = 3\left(\hat{a}^\dagger\right)^2$
   We can formulate a hypothesis for the n-th step as : $\left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] = n\left(\hat{a}^\dagger\right)^{n-1}$, applying the induction step for the $n+1$ case yields
   $$\begin{aligned}\left[\hat{a}, \left(\hat{a}^\dagger\right)^{n+1}\right] &= \left[\hat{a}, \hat{a}^\dagger\left(\hat{a}^\dagger\right)^n\right] = \hat{a}^\dagger\left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] + \left[\hat{a}, \hat{a}^\dagger\right]\left(\hat{a}^\dagger\right)^n \\ &= \hat{a}^\dagger n\left(\hat{a}^\dagger\right)^{n-1} + \left(\hat{a}^\dagger\right)^n = (n+1)\left(\hat{a}^\dagger\right)^n,\end{aligned} \tag{15}$$
   verifying the hypothesis.

2. We will follow the same strategy and use a proof by induction to obtain $\left[\hat{a}^\dagger, (\hat{a})^n\right] = -n\hat{a}^{n-1}$,
   For $n = 0:$ $\left[\hat{a}^\dagger, 1\right] = 0$
   For $n = 1:$ $\left[\hat{a}^\dagger, \hat{a}\right] = -1$
   For $n = 2:$ $\left[\hat{a}^\dagger, \hat{a}^2\right] = \hat{a}[\hat{a}^\dagger, \hat{a}] + [\hat{a}^\dagger, \hat{a}]\hat{a} = -2\hat{a}$
   For $n = 3:$ $\left[\hat{a}^\dagger, \hat{a}^3\right] = \ldots = -3\hat{a}^2$
   Similarly, the hypothesis for the n-th step is : $\left[\hat{a}^\dagger, \hat{a}^n\right] = -n\hat{a}^{n-1}$, and applying the induction step for the $n+1$ case yields
   $$\begin{aligned}\left[\hat{a}^\dagger, \hat{a}^{n+1}\right] &= \left[\hat{a}^\dagger, \hat{a}\hat{a}^n\right] = \hat{a}\left[\hat{a}^\dagger, \hat{a}^n\right] + \left[\hat{a}^\dagger, \hat{a}\right]\hat{a}^n \\ &= -\hat{a}n\hat{a}^{n-1} - \hat{a}^n = -(n+1)\hat{a}^n.\end{aligned} \tag{16}$$

3. Now let $f(x) = \sum_n f_n x^n$ be a well defined function for any value of x. We can develop f, use the bilinearity of the commutator and obtain

$$\left[\hat{a}, f\left(\hat{a}^\dagger\right)\right] = \left[\hat{a}, \sum_n f_n \left(\hat{a}^\dagger\right)^n\right] = \sum_n f_n \left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] = \sum_n f_n \cdot n \left(\hat{a}^\dagger\right)^{n-1} = \frac{\partial f\left(\hat{a}^\dagger\right)}{\partial \hat{a}^\dagger} \tag{17}$$

4. Similarly

$$\left[\hat{a}^\dagger, f\left(\hat{a}\right)\right] = \left[\hat{a}^\dagger, \sum_n f_n \hat{a}^n\right] = \sum_n f_n \left[\hat{a}^\dagger, \hat{a}^n\right] = \sum_n f_n \cdot (-n)\hat{a}^{n-1} = -\frac{\partial f\left(\hat{a}\right)}{\partial \hat{a}} \tag{18}$$

5. We define $f(\alpha) = e^{-\alpha\hat{A}}\hat{B}e^{\alpha\hat{A}}$ and we perform a Taylor expansion of $f(\alpha)$ near $\alpha = 0$ using the Taylor expansion of the exponentials. Recall the Taylor expansion of a function g as

$$g(\alpha) \cong g(0) + g'(0)\alpha + \frac{1}{2!}g''(0)\alpha^2 + \frac{1}{3!}g'''(0)\alpha^3 + \dots$$

Expanding the exponentials and collecting the terms gives

$$\begin{aligned}
f(\alpha) &\cong \left(1 - \alpha\hat{A} + \frac{\alpha^2}{2!}\hat{A}^2 - \frac{\alpha^3}{3!}\hat{A}^3 + \dots\right)\hat{B}\left(1 + \alpha\hat{A} + \frac{\alpha^2}{2!}\hat{A}^2 + \frac{\alpha^3}{3!}\hat{A}^3 + \dots\right) \\
&\cong \hat{B} - \alpha\left(\hat{A}\hat{B} - \hat{B}\hat{A}\right) + \frac{\alpha^2}{2!}\left(\hat{A}^2\hat{B} - \hat{A}\hat{B}\hat{A} + \hat{B}\hat{A}^2 - \hat{A}\hat{B}\hat{A}\right) + \mathcal{O}(\alpha^3) \\
&\cong \hat{B} - \alpha\left[\hat{A}, \hat{B}\right] + \frac{\alpha^2}{2!}\left([\hat{A}, \hat{A}\hat{B}] + [\hat{B}\hat{A}, \hat{A}]\right) + \mathcal{O}(\alpha^3) \\
&\cong \hat{B} - \alpha\left[\hat{A}, \hat{B}\right] + \frac{\alpha^2}{2!}\left[\hat{A}, \hat{A}\hat{B} - \hat{B}\hat{A}\right] + \mathcal{O}(\alpha^3) \\
&\cong \hat{B} - \alpha\left[\hat{A}, \hat{B}\right] + \frac{\alpha^2}{2!}\left[\hat{A}, [\hat{A}, \hat{B}]\right] + \mathcal{O}(\alpha^3)
\end{aligned} \tag{19}$$

6. Let us define $\hat{A} = -\hat{a}^\dagger\hat{a}$, we have $\left[\hat{A}, \hat{a}\right] = \hat{a}$ and using the result from 5, we obtain

$$\begin{aligned}
\left[\hat{a}, e^{-\alpha\hat{a}^\dagger\hat{a}}\right] &= \left[\hat{a}, e^{\alpha\hat{A}}\right] = \hat{a}e^{\alpha\hat{A}} - e^{\alpha\hat{A}}\hat{a} \\
&= e^{\alpha\hat{A}}\left(e^{-\alpha\hat{A}}\hat{a}e^{\alpha\hat{A}} - \hat{a}\right) \\
&= e^{\alpha\hat{A}}\left(\hat{a} - \alpha\left[\hat{A}, \hat{a}\right] + \frac{\alpha^2}{2!}\left[\hat{A}, [\hat{A}, \hat{a}]\right] + \dots - \hat{a}\right) \\
&= e^{\alpha\hat{A}}\left(\hat{a} - \alpha\hat{a} + \frac{\alpha^2}{2!}\hat{a} + \dots - \hat{a}\right) = e^{\alpha\hat{A}}\left(1 - \alpha + \frac{\alpha^2}{2!} + \dots - 1\right)\hat{a} \\
&= e^{\alpha\hat{A}}\left(e^{-\alpha} - 1\right)\hat{a} = (e^{-\alpha} - 1)e^{-\alpha\hat{a}^\dagger\hat{a}}\hat{a}
\end{aligned} \tag{20}$$

1.3 Solution: Quantized field properties: linear momentum

Let's consider a plane transverse EM wave of frequency ω with a wave-vector $\boldsymbol{k}$ and linear polarization $\boldsymbol{\epsilon}$ and write down the expressions for the quantized EM field

$$\hat{\mathbf{E}} = i\left(\frac{\hbar\omega}{2\epsilon_0 L^3}\right)^{1/2}\left[\hat{a}\boldsymbol{\epsilon}e^{i(\boldsymbol{kr}-\omega t)} - \hat{a}^\dagger\boldsymbol{\epsilon}e^{-i(\boldsymbol{kr}-\omega t)}\right] \tag{21}$$

and

$$\hat{\boldsymbol{H}} = i\left(\frac{\hbar|\boldsymbol{k}|}{2\epsilon_0 c L^3}\right)^{1/2}\left[\hat{a}\frac{\boldsymbol{k}\times\boldsymbol{\epsilon}}{|\boldsymbol{k}|}e^{i(\boldsymbol{kr}-\omega t)} - \hat{a}^\dagger\frac{\boldsymbol{k}\times\boldsymbol{\epsilon}}{|\boldsymbol{k}|}e^{i(\boldsymbol{kr}-\omega t)}\right] \tag{22}$$

Now we substitute these expessions into the expression for the Pointing vector and note the followin

- terms containing a spacial exponent of the form $e^{in\boldsymbol{kr}}$ where $n \neq 0$ will average out in the integration over $d^3\boldsymbol{r}$
- for a linearly polarized plane wave $[\boldsymbol{\epsilon} \times [\boldsymbol{k} \times \boldsymbol{\epsilon}]] = \boldsymbol{k}$
- $\omega = c|\boldsymbol{k}|$

and thus we are left with

$$\mathbf{P} = \epsilon_0 \int d^3r\, \mathbf{E} \times \mathbf{B} = -\int d^3r \frac{\hbar \boldsymbol{k}}{2L^3} \left[-\hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a}\right] = \hbar \boldsymbol{k} \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) \tag{23}$$

Note, that to include the rest of modes into consideration, it is sufficient to summ the individual contributions over $\boldsymbol{k}$ as any cross terms average out during the spacial integration. The field thus carries no zero point momentum, as for each pair of $\boldsymbol{k}$ and $-\boldsymbol{k}$ the $\frac{1}{2}$ term cancel out.

1.4 Solution: Quantization of an electrical LC circuit

(a) A conventional electric LC circuit consists of a capacitor and an inductor, and the total energy includes the energy in the capacitor (electric energy) and the energy in the inductor (magnetic energy). Hence, classical Hamiltonian can be written as

$$H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

It is possible to rewrite this Hamiltonian in terms of E and B (electric and magnetic field in the capacitor and inductor) using standard relations and definitions of capacitance and inductance:

$$Q = CV = CdE$$

$$\Phi = IL = ANB$$

$$H = \frac{Cd^2E^2}{2} + \frac{A^2N^2B^2}{2L}$$

(b) For quantization, one can introduce the following set of canonically conjugate variables and their proper commutation relations:

$$\hat{Q} = -iQ_{zpf}(\hat{a} - \hat{a}^\dagger)$$

$$\hat{\Phi} = \Phi_{zpf}(\hat{a} + \hat{a}^\dagger)$$

Note that this is not a unique choice of variables. ZPF (zero point fluctuation) coefficients are introduced here for the normalization, which also can be defined in different ways. For these variables to be canonically conjugate, the following commutation relations should be requested:

$$[\hat{a}, \hat{a}^\dagger] = 1$$

$$[\hat{\Phi}, \hat{Q}] = i\hbar$$

Now using the definitions of variables one can compute the Hamiltonian, canceling the terms aa and $a^\dagger a^\dagger$, and expand the second commutator. The final Hamiltonian is

$$\hat{H} = \hbar \frac{1}{\sqrt{LC}}(a^\dagger a + \frac{1}{2})$$

And ZPF coefficients are

$$Q_{zpf} = \sqrt{\frac{\hbar}{2Z_0}}$$

$$\Phi_{zpf} = \sqrt{\frac{\hbar Z_0}{2}}$$

Here, $Z_0 = \sqrt{\frac{L}{C}}$ is a value that is called characteristic impedance.

The Hamiltonian expressed via electric and magnetic fields undergoes the same procedure since it has the same mathematical form, but just different symbols, so there are just different variables in the final expressions.

(c) By straightforward expansion of the expression, one obtains the zero point fluctuation coefficients derived above, and immediately gets a proof that Heisenberg uncertainty is satisfied.

(d) It is equal to Q_{zpf} coefficient.

1.5 The Casimir Effect

1.5.1 Simplified treatment

a) Note that

$$\omega_k = c|k| = c\sqrt{k_x^2 + k_y^2 + k_z^2} = c\sqrt{\left(\frac{\pi}{L}n_x\right)^2 + \left(\frac{\pi}{L}n_y\right)^2 + \left(\frac{\pi}{a}n_z\right)^2}$$

So the total energy

$$E(L,a) = \frac{\hbar}{2}\sum_k{}' \omega_k + 2 \cdot \frac{\hbar}{2}\sum_k \omega_k$$

where $\sum'$ denotes the summation over k_x or k_y or $k_z = 0$.
b) When $L \to \infty$, we can neglect the lateral term. So

$$E(L,a) = 2\frac{\hbar}{2}c\sum_n \omega_n = \hbar c \sum_n \left(\frac{n^2\pi^2}{a^2}\right)^{1/2} = \frac{\hbar\pi c}{a}\sum_n n$$

c) When $a \to \infty$, we can regard $u = \frac{n\pi}{a}$ as a continuum variable. So

$$E(L,\infty) = \hbar c \lim_{a\to\infty} \sum_n \frac{n\pi}{a} = \hbar c \frac{a}{\pi}\int_0^\infty u \, \mathrm{d}u$$

Obviously, the potential writes

$$U = E(L,a) - E(L,\infty) = \frac{\hbar\pi c}{a}\left[\sum_n n - \frac{a^2}{\pi^2}\int_0^\infty u \, \mathrm{d}u\right]$$

d) To eliminate the divergence, we add an exponential term in the summation and the integrand.

$$U = \frac{\hbar\pi c}{a}\left[\sum_n n e^{-\alpha\pi n/a} - \frac{a^2}{\pi^2}\int_0^\infty u e^{-\alpha u}\,\mathrm{d}u\right]$$

To calculate the result, we let $u = \pi n/a$ and define $F(n) = ne^{-\alpha\pi n/a}$. Then apply Euler-Maclaurin formula on (64)

$$\begin{aligned}
U &= \frac{\hbar\pi c}{a}\left[\sum_n n e^{-\alpha\pi n/a} - \frac{a^2}{\pi^2}\int_0^\infty u e^{-\alpha u}\,\mathrm{d}u\right] \\
&= \frac{\hbar\pi c}{a}\left[\sum_n F(n) - \int_0^\infty F(x)\mathrm{d}x\right] \\
&= \frac{\hbar\pi c}{a}\left(\frac{F(0)+F(\infty)}{2} + \frac{1}{6}\frac{F'(\infty)-F'(0)}{2!} - \frac{1}{30}\frac{F^{(3)}(\infty)-F^{(3)}(0)}{4!} + \cdots\right) \\
&= -\frac{1}{12}\frac{\hbar\pi c}{a} - \frac{1}{720}\left(\frac{\alpha\pi}{a}\right)^2\frac{\hbar\pi c}{a} + \cdots \\
&= -\frac{1}{12}\frac{\hbar\pi c}{a} - \frac{1}{720}\frac{\alpha^2\pi^2\hbar c}{a^3} + O\left(\frac{1}{a^5}\right)
\end{aligned}$$

e) From (70) we can derive the force

$$\frac{\partial U}{\partial a} = \frac{\hbar\pi c}{12a^2} + \frac{\alpha\pi^2\hbar c}{240a^4} + O\left(\frac{1}{a^6}\right)$$

1.5.2 General treatment

a) Similarly we have

$$\begin{aligned}
E(L,a) &= \hbar c\sum_{n_x,n_y,n_z}\sqrt{k_x^2+k_y^2+(\pi n_z/a)^2} \\
&= \hbar c\sum_{k_x,k_y}\left(\frac{1}{2}\sqrt{k_x^2+k_y^2} + \sum_{n_z}\sqrt{k_x^2+k_y^2+(\pi n_z/a)^2}\right)
\end{aligned}$$

Likewise, we replace the summation over k_x, k_y by integrals of continuum k_x, k_y. So we can rewrite (67) into a polar integral by defining $\sqrt{k_x^2+k_y^2} = k$

$$E(L,a) = \hbar c\frac{L^2}{\pi^2}\frac{\pi}{2}\int_0^\infty\left(\frac{1}{2}k + \sum_{n_z}\sqrt{k^2+(\pi n_z/a)^2}\right)k\,\mathrm{d}k$$

When $a \to \infty$, we can regard k_z as a continuum variable. So

$$E(L,\infty) = \hbar c\frac{L^2}{\pi^2}\frac{\pi}{2}\frac{a}{\pi}\int_0^\infty\int_0^\infty\sqrt{k^2+k_z^2}k\,\mathrm{d}k\,\mathrm{d}k_z$$

By introducing $u = a^2k^2/\pi^2$, for $U(a)$ we have

$$\begin{aligned}
U(a) &= \frac{\hbar c L^2}{2\pi}\frac{\pi^3}{a^3}\left[\frac{1}{2}\int_0^\infty\sqrt{u}\,\mathrm{d}u + \sum_{n_z}\int_0^\infty\sqrt{u+n^2u}\,\mathrm{d}u - \int_0^\infty\int_0^\infty\sqrt{u+n^2}\,\mathrm{d}u\,\mathrm{d}n\right] \\
&= U(0)\left[\frac{1}{2}f(0) + \sum_n f(n) - \int_0^\infty f(\zeta)\mathrm{d}\zeta\right]
\end{aligned}$$

where $U(0) = \hbar c L^2\pi^2/2a^3, f(\zeta) = \int_0^\infty \mathrm{d}u\sqrt{u+\zeta^2}$
c) Make a replacement

$$f(\zeta) \rightarrow F(\zeta, \zeta_m) = \int_0^\infty \mathrm{d}u \sqrt{u+\zeta^2} g\left(\sqrt{u+\zeta^2}\right) / \zeta_m\Big)$$

Let $w = \zeta^2 + u$ and

$$F(\zeta, \zeta_m) = G(\zeta) = \int_{\zeta^2}^\infty w^{1/2} f\left(\sqrt{w}/\zeta_m\right) \mathrm{d}w$$

Then (70) becomes

$$U(a) = U(0)\left[\frac{1}{2}G(0) + \sum_n G(n) - \int_0^\infty G(\zeta)\mathrm{d}\zeta\right]$$

Apply Euler-Maclaurin formula to (72) then we have
$U(a) = U(0)\left[\frac{G(0)+G(\infty)}{2} + \frac{1}{6}\frac{G'(\infty)-G'(0)}{2!} - \frac{1}{30}\frac{G^{(3)}(\infty)-G^{(3)}(0)}{4!} + \cdots\right]$
Obviously the first term in (73) vanishes since G vanishes at 0 and ∞. The second term also vanishes because the derivative of G takes the form of $\zeta f(\zeta/\zeta_m)$. The third term with second-order derivatives survives. So the potential

$$U(a) = k\frac{1}{a^3} + O\left(\frac{1}{a^5}\right)$$

where k is a negative coeefficient.

d) Obviously, from (74) we find that $\partial U/\partial a$ takes + sign and is proportional to $1/a^4$ if we neglect the higher order terms.